Propositional Logic

Statement (Proposition)

A *Statement* is a sentence that is either **True** or **False**

Examples:

2 + 2 = 4 True

3 x 3 = 8 False

787009911 is a prime

Non-examples:

$x+y>0$

$x^2+y^2=z^2$

They are true for some values of x and y
but are false for some other values of x and y.

Logic Operators

$\neg$ ::= NOT

~p is true if p is false

$\wedge$ ::= AND

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$\vee$ ::= OR

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Compound Statement

p = "it is hot"

q = "it is sunny"

It is hot and sunny	$p \wedge q$
It is not hot but sunny	$\neg p \wedge q$
It is neither hot nor sunny	$\neg p \wedge \neg q$

Exclusive-Or

coffee "or" tea ← $\oplus$ exclusive-or

How to construct a compound statement for exclusive-or?

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Idea 1: Look at the true rows

Want the formula to be true exactly when the input belongs to a "true" row.

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

The input is the second row exactly if this sub-formula is satisfied

And the formula is true exactly when the input is the second row or the third row.

Exclusive-Or

coffee "or" tea ← $\oplus$ exclusive-or

How to construct a compound statement for exclusive-or?

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Idea 2: Look at the false rows

Want the formula to be true exactly when the input does **not** belong to a "false" row.

$$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$$

The input is the first row exactly if this sub-formula is satisfied

And the formula is true exactly when the input is not in the 1st row and the 4th row.

Logical Equivalence

Idea 3: Guess and check

$$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$$

p	q	$p \oplus q$	$p \vee q$	$\neg(p \wedge q)$	
T	T	F	T	F	F
T	F	T	T	T	T
F	T	T	T	T	T
F	F	F	F	T	F

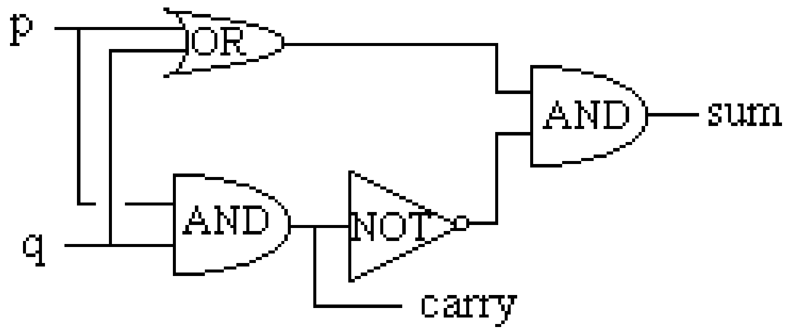
Logical equivalence: Two statements have the same truth table

As you see, there are many different ways to write the same logical formula.

One can always use a truth table to check whether two statements are equivalent.

Writing Logical Formula for a Truth Table

Digital logic:



p	q	sum	carry
1	1	0	1
1	0	1	0
0	1	1	0
0	0	0	0

Given a digital circuit, we can construct the truth table.

Now, suppose we are given only the truth table (i.e. the specification), how can we construct a circuit (i.e. formula) that has the same function?

Writing Logical Formula for a Truth Table

Use idea 1 or idea 2.

Idea 1: Look at the true rows and take the "or".

	p	q	r	output
$p \wedge q \wedge r$	T	T	T	F
$p \wedge q \wedge \neg r$	T	T	F	T
$p \wedge \neg q \wedge r$	T	F	T	T
$p \wedge \neg q \wedge \neg r$	T	F	F	F
$\neg p \wedge q \wedge r$	F	T	T	T
$\neg p \wedge q \wedge \neg r$	F	T	F	T
$\neg p \wedge \neg q \wedge r$	F	F	T	T
$\neg p \wedge \neg q \wedge \neg r$	F	F	F	F

$$(p \wedge q \wedge \neg r)$$
$$\vee (p \wedge \neg q \wedge r)$$
$$\vee (\neg p \wedge q \wedge r)$$
$$\vee (\neg p \wedge q \wedge \neg r)$$
$$\vee (\neg p \wedge \neg q \wedge r)$$

The formula is true exactly when the input is one of the true rows.

Writing Logical Formula for a Truth Table

Idea 2: Look at the false rows, negate and take the "and".

	p	q	r	output	
$p \wedge q \wedge r$	T	T	T	F	$\neg(p \wedge q \wedge r)$
$p \wedge q \wedge \neg r$	T	T	F	T	
$p \wedge \neg q \wedge r$	T	F	T	T	
$p \wedge \neg q \wedge \neg r$	T	F	F	F	$\wedge \neg(p \wedge \neg q \wedge \neg r)$
$\neg p \wedge q \wedge r$	F	T	T	T	
$\neg p \wedge q \wedge \neg r$	F	T	F	T	
$\neg p \wedge \neg q \wedge r$	F	F	T	T	
$\neg p \wedge \neg q \wedge \neg r$	F	F	F	F	$\wedge \neg(\neg p \wedge \neg q \wedge \neg r)$

The formula is true exactly when the input is not one of the false row.

DeMorgan's Laws

Logical equivalence: Two statements have the same truth table

De Morgan's Law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Statement: Tom is in the football team and the basketball team.

Negation: Tom is not in the football team or not in the basketball team.

De Morgan's Law

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Statement: The number 783477841 is divisible by 7 or 11.

Negation: The number 783477841 is not divisible by 7 and not divisible by 11.

DeMorgan's Laws

Logical equivalence: Two statements have the same truth table

De Morgan's Law $$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

De Morgan's Law $$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Simplifying Statement

$$\neg(\neg p \wedge q) \wedge (p \vee q)$$

$$\equiv (\neg\neg p \vee \neg q) \wedge (p \vee q)$$ DeMorgan

$$\equiv (p \vee \neg q) \wedge (p \vee q)$$

$$\equiv p \vee (\neg q \wedge q)$$ Distributive

$$\equiv p \vee \text{False}$$

$$\equiv p$$

See textbook for more identities.

Tautology, Contradiction

A tautology is a statement that is always true.

$$p \vee \neg p$$

$$(p \wedge q) \vee (\neg q \wedge p) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q)$$

A contradiction is a statement that is always false. (negation of a tautology)

$$p \wedge \neg p$$

$$(p \vee q) \wedge (\neg q \vee p) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee q)$$

$$((p \wedge r) \vee (q \wedge r)) \wedge (\neg(p \vee q) \vee r)$$

In general it is "difficult" to tell whether a statement is a contradiction.
It is one of the most important problems in CS – the satisfiability problem.

Quick Summary

Key points to know.

1. Write a logical formula from a truth table.
2. Check logical equivalence of two logical formulas.
3. DeMorgan's rule and other simple logical rules (e.g. distributive).
4. Use simple logical rules to simplify a logical formula.

Conditional Statement

If p then q

$$p \longrightarrow q$$

p is called the hypothesis; q is called the conclusion

The department says: "If your GPA is 4.0, then you don't need to pay tuition fee."

When is the above sentence false?

- It is false when your GPA is 4.0 but you still have to pay tuition fee.
- But it is not false if your GPA is below 4.0.

Another example: "If there is a bandh today, then there is no class."

When is the above sentence false?

Logic Operator

$$\rightarrow ::= \text{IMPLIES}$$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Convention: if we don't say anything wrong, then it is not false, and thus true.

Logical Equivalence

$$p \rightarrow q \equiv ?$$

If you see a question in the above form, there are usually 3 ways to deal with it.

(1) Truth table

(2) Use logical rules

(3) Intuition

If-Then as Or

$$p \longrightarrow q \equiv ?$$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Idea 2: Look at the false rows, negate and take the **"and"**.

$$\neg(P \wedge \neg Q)$$
$$\equiv \neg P \vee Q$$

- If you don't give me all your money, then I will kill you.
- Either you give me all your money or I will kill you (or both).

- If you talk to her, then you can never talk to me.
- Either you don't talk to her or you can never talk to me (or both).

Negation of If-Then

$$\neg(p \rightarrow q) \equiv ?$$

- If your GPA is 4.0, then you don't need to pay tuition fee.
- Your term GPA is 4.0 and you still need to pay tuition fee.

- If my computer is not working, then I cannot finish my homework.
- My computer is not working but I can finish my homework.

$$\begin{aligned} &\neg(P \rightarrow Q) \\ \equiv\ &\neg(\neg P \vee Q) && \text{previous slide} \\ \equiv\ &\neg\neg P \wedge \neg Q && \text{DeMorgan} \\ \equiv\ &P \wedge \neg Q \end{aligned}$$

Contrapositive

The contrapositive of "if p then q" is "if ~q then ~p".

Statement: If you are a CS year 1 student,
then you are taking CTS002.

Contrapositive: If you are not taking CTS002,
then you are not a CS year 1 student.

Statement: If x^2 is an even number,
then x is an even number.

Contrapositive: If x is an odd number,
then x^2 is an odd number.

Fact: A conditional statement is logically equivalent to its contrapositive.

Proofs

Statement: If P, then Q

Contrapositive: If ¬Q, then ¬P.

P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$$P \rightarrow Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg Q \rightarrow \neg P$$

If, Only-If

- You will succeed if you work hand.
- You will succeed only if you work hard.

R if S means "**if S then R**" or equivalently "**S implies R**"

We also say S is a sufficient condition for R.

R only if S means "**if R then S**" or equivalently "**R implies S**"

We also say S is a necessary condition for R.

You will succeed if and only if you work hard.

P if and only if (iff) Q means P and Q are logically equivalent.

That is, P implies Q and Q implies P.

Math vs English

Parent: if you don't clean your room, then you can't watch a DVD.

C D

This sentence says $\neg C \rightarrow \neg D$

In real life it also means $C \rightarrow D$

So $C \leftrightarrow D$

Mathematician: if a number x greater than 2 is not an odd number, then x is not a prime number.

This sentence says $\neg O \rightarrow \neg P$

But of course it doesn't mean $O \rightarrow P$

Necessary, Sufficient Condition

Mathematician: if a number x greater than 2 is not an odd number,
then x is not a prime number.

This sentence says $\neg O \to \neg P$

But of course it doesn't mean $O \to P$

Being an odd number > 2 is a necessary condition for this number to be prime.

Being a prime number > 2 is a sufficient condition for this number to be odd.

Necessary AND Sufficient Condition

$$\leftrightarrow \ ::= \text{IFF}$$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: $P \longleftrightarrow Q$ is equivalent to $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$

Note: $P \longleftrightarrow Q$ is equivalent to $(P \longrightarrow Q) \wedge (\neg P \longrightarrow \neg Q)$

Is the statement "x is an even number if and only if x^2 is an even number" true?