LECTURE NOTES

on

PROBABILITY and STATISTICS

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SAMPLE SPACES

DEFINITION :

The *sample space* is the set of all possible outcomes of an experiment.

EXAMPLE : When we *flip a coin* then sample space is

$$\mathcal{S} \ = \ \{ \ H \ , \ T \ \} \ ,$$

where

H denotes that the coin lands "Heads up"

and

T denotes that the coin lands "Tails up".

For a "*fair coin* " we expect H and T to have the same "*chance* " of occurring, *i.e.*, if we flip the coin many times then about 50 % of the outcomes will be H.

We say that the *probability* of H to occur is 0.5 (or 50 %) .

The probability of T to occur is then also 0.5.

EXAMPLE :

When we *roll a fair die* then the sample space is

$$\mathcal{S} \ = \ \{ \ 1 \ , \ 2 \ , \ 3 \ , \ 4 \ , \ 5 \ , \ 6 \ \} \ .$$

The probability the die lands with k up is $\frac{1}{6}$, $(k = 1, 2, \cdots, 6)$.

When we roll it 1200 times we expect a 5 up about 200 times.

The probability the die lands with an *even number* up is

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} \ = \ \frac{1}{2} \ .$$

EXAMPLE :

When we toss a coin 3 times and record the results in the *sequence* that they occur, then the sample space is

$$\mathcal{S} = \{ HHH , HHT , HTH , HTT , THH , THT , TTH , TTT \} .$$

Elements of $\mathcal{S}$ are "*vectors*", "*sequences*", or "*ordered outcomes*".

We may expect each of the 8 outcomes to be equally likely.

Thus the probability of the sequence HTT is $\frac{1}{8}$.

The probability of a sequence to contain precisely two Heads is

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} \quad = \quad \frac{3}{8} .$$

EXAMPLE : When we toss a coin 3 times and record the results without paying attention to the order in which they occur, *e.g.*, if we only record the number of Heads, then the sample space is

$$\mathcal{S} \;=\; \Big\{ \{H,H,H\} \;,\; \{H,H,T\} \;,\; \{H,T,T\} \;,\; \{T,T,T\} \Big\} \;.$$

The outcomes in $\mathcal{S}$ are now *sets* ; *i.e.*, order is not important.

Recall that the ordered outcomes are

$$\{ \; HHH \;,\; HHT \;,\; HTH \;,\; HTT \;,\; THH \;,\; THT \;,\; TTH \;,\; TTT \; \} \;.$$

Note that

$\{H,H,H\}$	corresponds to	*one*	of the ordered outcomes,
$\{H,H,T\}$	,,	*three*	,,
$\{H,T,T\}$	,,	*three*	,,
$\{T,T,T\}$	,,	*one*	,,

Thus $\{H,H,H\}$ and $\{T,T,T\}$ each occur with probability $\frac{1}{8}$,

while $\{H,H,T\}$ and $\{H,T,T\}$ each occur with probability $\frac{3}{8}$.

Events

In Probability Theory subsets of the sample space are called *events*.

EXAMPLE : The set of basic outcomes of rolling a die *once* is

$$\mathcal{S} \ = \ \{ \, 1 \, , \ 2 \, , \ 3 \, , \ 4 \, , \ 5 \, , \ 6 \, \} \, ,$$

so the subset $E \ = \ \{ \, 2 \, , \ 4 \, , \ 6 \, \}$ is an example of an event.

If a die is rolled *once* and it lands with a 2 *or* a 4 *or* a 6 up then we say that the event E has *occurred*.

We have already seen that the probability that E occurs is

$$P(E) \ = \ \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \ = \ \frac{1}{2} \, .$$

The Algebra of Events

Since events are *sets*, namely, subsets of the sample space $\mathcal{S}$, we can do the usual *set operations* :

If E and F are events then we can form

E^c	the *complement* of E
$E \cup F$	the *union* of E and F
EF	the *intersection* of E and F

We write $E \subset F$ if E is a *subset* of F.

REMARK : In Probability Theory we use

E^c	instead of $\bar{E}$,
EF	instead of $E \cap F$,
$E \subset F$	instead of $E \subseteq F$.

If the sample space $\mathcal{S}$ is *finite* then we typically allow any subset of $\mathcal{S}$ to be an event.

EXAMPLE : If we randomly draw *one character* from a box containing the characters a, b, and c, then the sample space is

$$\mathcal{S} \;=\; \{a \;,\; b \;,\; c\} \;,$$

and there are 8 possible events, namely, those in the set of events

$$\mathcal{E} \;=\; \Big\{ \;\{\;\}\;,\; \{a\}\;,\; \{b\}\;,\; \{c\}\;,\; \{a,b\}\;,\; \{a,c\}\;,\; \{b,c\}\;,\; \{a,b,c\}\; \Big\} \;.$$

If the outcomes a, b, and c, are equally likely to occur, then

$$P(\{\;\}) = 0 \quad,\quad P(\{a\}) = \frac{1}{3} \quad,\quad P(\{b\}) = \frac{1}{3} \quad,\quad P(\{c\}) = \frac{1}{3} \;,$$

$$P(\{a,b\}) = \frac{2}{3} \;,\; P(\{a,c\}) = \frac{2}{3} \;,\; P(\{b,c\}) = \frac{2}{3} \;,\; P(\{a,b,c\}) = 1 \;.$$

For example, $P(\{a,b\})$ is the probability the character is an a *or* a b.

We always assume that the set $\mathcal{E}$ of allowable events *includes the complements, unions, and intersections* of its events.

EXAMPLE : If the sample space is

$$\mathcal{S} = \{a \,,\, b \,,\, c \,,\, d\} \,,$$

and we start with the events

$$\mathcal{E}_0 = \Big\{ \{a\} \,,\, \{c, d\} \Big\} \,,$$

then this set of events needs to be extended to (at least)

$$\mathcal{E} = \Big\{ \{\,\} \,,\, \{a\} \,,\, \{c, d\} \,,\, \{b, c, d\} \,,\, \{a, b\} \,,\, \{a, c, d\} \,,\, \{b\} \,,\, \{a, b, c, d\} \Big\} \,.$$

EXERCISE : Verify $\mathcal{E}$ includes complements, unions, intersections.

Axioms of Probability

A *probability function* P assigns a real number (the *probability* of E) to every event E in a sample space $\mathcal{S}$.

$P(\cdot)$ must satisfy the following basic properties :

- $0 \leq P(E) \leq 1$,

- $P(\mathcal{S}) = 1$,

- For any *disjoint events* E_i , $i = 1, 2, \cdots, n$, we have

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots P(E_n) .$$

Further Properties

PROPERTY 1 :

$$P(E \cup E^c) \;=\; P(E) \;+\; P(E^c) \;=\; 1 \,. \qquad (\text{ Why ? })$$

Thus

$$P(E^c) \;=\; 1 \;-\; P(E) \,.$$

EXAMPLE :

What is the probability of at least one "H" in *four tosses* of a coin?

SOLUTION : The sample space $\mathcal{S}$ will have 16 outcomes. (Which?)

$$P(\text{at least one H}) \;=\; 1 \;-\; P(\text{no H}) \;=\; 1 \;-\; \frac{1}{16} \;=\; \frac{15}{16} \,.$$

PROPERTY 2 :

$$P(E \cup F) \ = \ P(E) \ + \ P(F) \ - \ P(EF) \ .$$

PROOF (using the third axiom) :

$$P(E \cup F) \ = \ P(EF) \ + \ P(EF^c) \ + \ P(E^cF)$$

$$= \ [P(EF) \ + \ P(EF^c)] \ + \ [P(EF) \ + \ P(E^cF)] \ - \ P(EF)$$

$= \mathrm{P(E)} \ + \ \mathrm{P(F)}$ - P(EF) . (Why ?)

NOTE :

- Draw a Venn diagram with E and F to see this !
- The formula is similar to the one for the number of elements :

$$n(E \cup F) \ = \ n(E) \ + \ n(F) \ - \ n(EF) \ .$$

So far our sample spaces $\mathcal{S}$ have been *finite*.

$\mathcal{S}$ can also be *countably infinite*, *e.g.*, the set $\mathbb{Z}$ of all integers.

$\mathcal{S}$ can also be *uncountable*, *e.g.*, the set $\mathbb{R}$ of all real numbers.

EXAMPLE : Record the low temperature in Montreal on January 8 in each of a large number of years.

We can take $\mathcal{S}$ to be the set of *all real numbers*, *i.e.*, $\mathcal{S} = \mathbb{R}$.

(Are there are other choices of $\mathcal{S}$?)

What probability would you expect for the following *events* to have?

(a) $P(\{\pi\})$ (b) $P(\{x \ : \ -\pi < x < \pi\})$

(How does this differ from finite sample spaces?)

We will encounter such infinite sample spaces many times $\cdots$

Counting Outcomes

We have seen examples where the outcomes in a *finite* sample space $\mathcal{S}$ are *equally likely* , *i.e.*, they have *the same probability* .

Such sample spaces occur quite often.

Computing probabilities then requires counting *all* outcomes and counting *certain types* of outcomes .

The counting has to be done carefully!

We will discuss a number of representative examples in detail.

Concepts that arise include *permutations* and *combinations*.

Permutations

- Here we count of the number of "*words*" that can be formed from a collection of items (*e.g.*, letters).

- (Also called *sequences* , *vectors* , *ordered sets* .)

- The *order* of the items in the word is important;

  e.g., the word *acb* is different from the word *bac* .

- The *word length* is the number of characters in the word.

NOTE :

For *sets* the order is not important. For example, the set $\{a,c,b\}$ is the same as the set $\{b,a,c\}$.

EXAMPLE : Suppose that four-letter words of *lower case* alphabetic characters are generated randomly with equally likely outcomes. (Assume that *letters may appear repeatedly*.)

(a) How many four-letter words are there in the sample space $\mathcal{S}$?

SOLUTION : $26^4 = 456,976$.

(b) How many four-letter words are there are there in $\mathcal{S}$ that start with the letter "s" ?

SOLUTION : 26^3 .

(c) What is the *probability* of generating a four-letter word that starts with an "s" ?

SOLUTION :

$$\frac{26^3}{26^4} = \frac{1}{26} \cong 0.038 .$$

Could this have been computed more easily?

EXAMPLE : How many re-orderings (*permutations*) are there of the string *abc* ? (Here *letters may appear only once*.)

SOLUTION : Six, namely, *abc* , *acb* , *bac* , *bca* , *cab* , *cba* .

If these permutations are generated randomly with equal probability then what is the probability the word starts with the letter " *a* " ?

SOLUTION :

$$\frac{2}{6} = \frac{1}{3} .$$

EXAMPLE : In general, if the word length is n and *all characters are distinct* then there are $n!$ permutations of the word. (Why ?)

If these permutations are generated randomly with equal probability then what is the probability the word starts with a particular letter ?

SOLUTION :

$$\frac{(n-1)!}{n!} = \frac{1}{n} . \qquad (\text{ Why ? })$$

EXAMPLE : How many

words of length k

can be formed from

a set of n (distinct) characters ,

(where $k \leq n$) ,

when letters can be used *at most once* ?

SOLUTION :

$$\begin{aligned} & n\,(n-1)\,(n-2)\,\cdots\,(n-(k-1)) \\ = \;& n\,(n-1)\,(n-2)\,\cdots\,(n-k+1) \\ = \;& \frac{n!}{(n-k)!} \qquad (\text{ Why ? }) \end{aligned}$$

EXAMPLE : *Three-letter words* are generated randomly from the *five* characters a , b , c , d , e , where letters can be *used at most once.*

(a) How many three-letter words are there in the sample space $\mathcal{S}$?

SOLUTION : $5 \cdot 4 \cdot 3 \ = \ 60$.

(b) How many words containing a , b are there in $\mathcal{S}$?

SOLUTION : First place the characters

$$a \ , \ b$$

i.e., *select the two indices* of the locations to place them.

This can be done in

$$3 \ \times \ 2 \ = \ 6 \ \text{ ways} . \qquad (\text{ Why ? })$$

There remains one position to be filled with a c , d or an e .

Therefore the number of words is $3 \ \times \ 6 \ = \ 18$.

(c) Suppose the 60 solutions in the sample space are *equally likely* .

What is the *probability* of generating a three-letter word that contains the letters a and b ?

SOLUTION :

$$\frac{18}{60} \quad = \quad 0.3 \; .$$

EXERCISE :

Suppose the sample space $\mathcal{S}$ consists of all *five-letter* words having *distinct alphabetic characters* .

- How many words are there in $\mathcal{S}$?

- How many "special" words are in $\mathcal{S}$ for which *only* the second and the fourth character are vowels, *i.e.*, one of $\{a, e, i, o, u, y\}$?

- Assuming the outcomes in $\mathcal{S}$ to be equally likely, what is the probability of drawing such a special word?

Combinations

Let S be a set containing n (distinct) elements.

Then

a *combination* of k elements from S ,

is

any selection of k elements from S ,

where *order is not important* .

(Thus the selection is a *set* .)

NOTE : By definition a *set* always has *distinct elements* .

EXAMPLE :

There are three *combinations* of 2 elements chosen from the set

$$S = \{a\ ,\ b\ ,\ c\}\ ,$$

namely, the *subsets*

$$\{a,b\}\quad,\quad\{a,c\}\quad,\quad\{b,c\}\quad,$$

whereas there are six *words* of 2 elements from S ,

namely,

$$ab\ ,\ ba\quad,\quad ac\ ,\ ca\quad,\quad bc\ ,\ cb\ .$$

In general, given

$$\text{a set} \quad S \quad \text{of} \quad n \quad \text{elements} \ ,$$

the number of possible subsets of k elements from S equals

$$\binom{n}{k} \quad \equiv \quad \frac{n!}{k! \ (n-k)!} \ .$$

REMARK : The notation $\binom{n}{k}$ is referred to as

”n *choose* k ”.

NOTE : $$\binom{n}{n} \quad = \quad \frac{n!}{n! \ (n-n)!} \quad = \quad \frac{n!}{n! \ \ 0!} \quad = \quad 1 \ ,$$

since $0! \ \equiv \ 1$ (by “convenient definition” !) .

PROOF :

First recall that there are

$$n\ (n-1)\ (n-2)\ \cdots\ (n-k+1) \quad = \quad \frac{n!}{(n-k)!}$$

possible *sequences* of k distinct elements from S .

However, every sequence of length k has $k!$ permutations of itself, and each of these defines the same subset of S.

Thus the total number of subsets is

$$\frac{n!}{k!\ (n-k)!} \quad \equiv \quad \binom{n}{k} \ .$$

EXAMPLE :

In the previous example, with 2 elements chosen from the set

$$\{a \ , \ b \ , \ c\} \ ,$$

we have $n = 3$ and $k = 2$, so that there are

$$\frac{3!}{(3-2)!} \quad = \quad 6 \quad \textit{words} \ ,$$

namely

$$ab \ , \ ba \quad , \quad ac \ , \ ca \quad , \quad bc \ , \ cb \quad ,$$

while there are

$$\binom{3}{2} \quad \equiv \quad \frac{3!}{2! \ (3-2)!} \quad = \quad \frac{6}{2} \quad = \quad 3 \quad \textit{subsets} \ ,$$

namely

$$\{a,b\} \quad , \quad \{a,c\} \quad , \quad \{b,c\} \ .$$

EXAMPLE : If we choose 3 elements from $\{a\ ,\ b\ ,\ c\ ,\ d\}$, then

$$n \ = \ 4 \ \text{ and } \ k \ = \ 3 \ ,$$

so there are

$$\frac{4!}{(4-3)!} \ = \ 24 \quad \text{words, namely} \ :$$

$$\begin{array}{ccccccc} abc & , & abd & , & acd & , & bcd \ , \\ acb & , & adb & , & adc & , & bdc \ , \\ bac & , & bad & , & cad & , & cbd \ , \\ bca & , & bda & , & cda & , & cdb \ , \\ cab & , & dab & , & dac & , & dbc \ , \\ cba & , & dba & , & dca & , & dcb \ , \end{array}$$

while there are

$$\binom{4}{3} \ \equiv \ \frac{4!}{3!\ (4-3)!} \ = \ \frac{24}{6} \ = \ 4 \quad \text{subsets} \ ,$$

namely,

$$\{a, b, c\} \ , \ \{a, b, d\} \ , \ \{a, c, d\} \ , \ \{b, c, d\} \ .$$

EXAMPLE :

(a) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if order is not important?

SOLUTION :

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = 210 \ .$$

(b) If each of these 210 outcomes is equally likely then what is the probability that a particular person is on the committee?

SOLUTION :

$$\binom{9}{3} / \binom{10}{4} = \frac{84}{210} = \frac{4}{10} \ . \qquad (\text{ Why ? })$$

Is this result surprising?

(c) What is the probability that a particular person is *not* on the committee?

SOLUTION :

$$\binom{9}{4} / \binom{10}{4} \quad = \quad \frac{126}{210} \quad = \quad \frac{6}{10} \ . \qquad\qquad (\text{ Why ? })$$

Is this result surprising?

(d) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if one is to be the chairperson?

SOLUTION :

$$\binom{10}{1} \binom{9}{3} \quad = \quad 10 \binom{9}{3} \quad = \quad 10 \ \frac{9!}{3! \ (9-3)!} \quad = \quad 840 \ .$$

QUESTION : Why is this four times the number in (a) ?

EXAMPLE : *Two balls* are selected at random from a bag with *four white* balls and *three black* balls, where order is not important.

What would be an appropriate sample space $\mathcal{S}$?

SOLUTION : Denote the set of balls by

$$B = \{w_1 , w_2 , w_3 , w_4 , b_1 , b_2 , b_3\} ,$$

where same color balls are made "distinct" by numbering them.

Then a good choice of the sample space is

$$\mathcal{S} = \text{the set of } \textit{all subsets} \text{ of } \textit{two balls} \text{ from } B ,$$

because the wording "*selected at random*" suggests that each such subset has the same chance to be selected.

The number of outcomes in $\mathcal{S}$ (which are sets of two balls) is then

$$\binom{7}{2} = 21 .$$

EXAMPLE : (continued $\cdots$)

(*Two balls* are selected at random from a bag with *four white* balls and *three black* balls.)

- What is the probability that both balls are white?

SOLUTION :

$$\binom{4}{2} / \binom{7}{2} = \frac{6}{21} = \frac{2}{7}.$$

- What is the probability that both balls are black?

SOLUTION :

$$\binom{3}{2} / \binom{7}{2} = \frac{3}{21} = \frac{1}{7}.$$

- What is the probability that one is white and one is black?

SOLUTION :

$$\binom{4}{1} \binom{3}{1} / \binom{7}{2} = \frac{4 \cdot 3}{21} = \frac{4}{7}.$$

(Could this have been computed differently?)

EXAMPLE : (continued $\cdots$)

In detail, the sample space $\mathcal{S}$ is

$$
\left\{ \begin{array}{cccccccc}
\{w_1,w_2\}, & \{w_1,w_3\}, & \{w_1,w_4\}, & | & \{w_1,b_1\}, & \{w_1,b_2\}, & \{w_1,b_3\}, & \\
 & \{w_2,w_3\}, & \{w_2,w_4\}, & | & \{w_2,b_1\}, & \{w_2,b_2\}, & \{w_2,b_3\}, & \\
 & & \{w_3,w_4\}, & | & \{w_3,b_1\}, & \{w_3,b_2\}, & \{w_3,b_3\}, & \\
 & & & | & \{w_4,b_1\}, & \{w_4,b_2\}, & \{w_4,b_3\}, & \\
 & & & & \overline{\phantom{\{w_4,b_1\}}} & \overline{\phantom{\{w_4,b_2\}}} & \overline{\phantom{\{w_4,b_3\}}} & \\
 & & & & & \{b_1,b_2\}, & \{b_1,b_3\}, & \\
 & & & & & & \{b_2,b_3\} &
\end{array} \right\}
$$

- $\mathcal{S}$ has 21 outcomes, *each of which is a set* .
- We assumed each outcome of $\mathcal{S}$ has probability $\frac{1}{21}$.
- The *event* "both balls are white" contains 6 outcomes.
- The *event* "both balls are black" contains 3 outcomes.
- The *event* "one is white and one is black" contains 12 outcomes.
- What would be different had we worked with *sequences* ?

EXERCISE :

Three balls are selected at random from a bag containing

2 *red* , 3 *green* , 4 *blue* balls .

What would be an appropriate sample space $\mathcal{S}$?

What is the the number of outcomes in $\mathcal{S}$?

What is the probability that all three balls are *red* ?

What is the probability that all three balls are *green* ?

What is the probability that all three balls are *blue* ?

What is the probability of *one red*, *one green*, and *one blue* ball ?

EXAMPLE : A bag contains 4 *black* balls and 4 *white* balls.

Suppose one draws *two balls at the time*, until the bag is empty.

What is the probability that each drawn pair is *of the same color*?

SOLUTION : An *example of an outcome* in the sample space $\mathcal{S}$ is

$$\Big\{ \ \{w_1, w_3\} \ , \ \{w_2, b_3\} \ , \ \{w_4, b_1\} \ , \ \{b_2, b_4\} \ \Big\} \ .$$

The number of such *doubly unordered* outcomes in $\mathcal{S}$ is

$$\frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{1}{4!}\,\frac{8!}{2!\,6!}\,\frac{6!}{2!\,4!}\,\frac{4!}{2!\,2!}\,\frac{2!}{2!\,0!} = \frac{1}{4!}\,\frac{8!}{(2!)^4} = 105 \quad \text{(Why?)}$$

The number of such outcomes with *pairwise the same color* is

$$\frac{1}{2!}\binom{4}{2}\binom{2}{2} \ \cdot \ \frac{1}{2!}\binom{4}{2}\binom{2}{2} \quad = \quad 3 \ \cdot \ 3 \quad = \quad 9 \ . \qquad (\text{ Why ? })$$

Thus the probability each pair is *of the same color* is $9/105 \ = \ 3/35$.

EXAMPLE : (continued $\cdots$)

The 9 outcomes of *pairwise the same color* constitute the *event*

$$\Big\{ \quad \big\{ \{w_1,w_2\}\ ,\ \{w_3,w_4\}\ ,\ \{b_1,b_2\}\ ,\ \{b_3,b_4\} \big\}\ ,$$
$$\big\{ \{w_1,w_3\}\ ,\ \{w_2,w_4\}\ ,\ \{b_1,b_2\}\ ,\ \{b_3,b_4\} \big\}\ ,$$
$$\big\{ \{w_1,w_4\}\ ,\ \{w_2,w_3\}\ ,\ \{b_1,b_2\}\ ,\ \{b_3,b_4\} \big\}\ ,$$

$$\big\{ \{w_1,w_2\}\ ,\ \{w_3,w_4\}\ ,\ \{b_1,b_3\}\ ,\ \{b_2,b_4\} \big\}\ ,$$
$$\big\{ \{w_1,w_3\}\ ,\ \{w_2,w_4\}\ ,\ \{b_1,b_3\}\ ,\ \{b_2,b_4\} \big\}\ ,$$
$$\big\{ \{w_1,w_4\}\ ,\ \{w_2,w_3\}\ ,\ \{b_1,b_3\}\ ,\ \{b_2,b_4\} \big\}\ ,$$

$$\big\{ \{w_1,w_2\}\ ,\ \{w_3,w_4\}\ ,\ \{b_1,b_4\}\ ,\ \{b_2,b_3\} \big\}\ ,$$
$$\big\{ \{w_1,w_3\}\ ,\ \{w_2,w_4\}\ ,\ \{b_1,b_4\}\ ,\ \{b_2,b_3\} \big\}\ ,$$
$$\big\{ \{w_1,w_4\}\ ,\ \{w_2,w_3\}\ ,\ \{b_1,b_4\}\ ,\ \{b_2,b_3\} \big\} \quad \Big\}\ .$$

EXERCISE :

- How many ways are there to choose a committee of 4 persons from a group of 6 persons, if order is not important?
- Write down the list of all these possible committees of 4 persons.
- If each of these outcomes is equally likely then what is the probability that two particular persons are on the committee?

EXERCISE :

Two balls are selected at random from a bag with three white balls and two black balls.

- Show all elements of a suitable sample space.
- What is the probability that both balls are white?

EXERCISE :

We are interested in *birthdays* in a class of 60 students.

- What is a good sample space $\mathcal{S}$ for this purpose?

- How many outcomes are there in $\mathcal{S}$?

- What is the probability of *no common birthdays* in this class?

- What is the probability of *common birthdays* in this class?

EXAMPLE :

How many *nonnegative* integer solutions are there to

$$x_1 \ + \ x_2 \ + \ x_3 \quad = \quad 17 \ ?$$

SOLUTION :

Consider seventeen 1's separated by bars to indicate the possible values of x_1, x_2, and x_3, *e.g.*,

$$111|111111111|11111 \ .$$

The total number of positions in the "display" is $17 + 2 = 19$.

The total number of *nonnegative* solutions is now seen to be

$$\binom{19}{2} \quad = \quad \frac{19!}{(19-2)! \ 2!} \quad = \quad \frac{19 \times 18}{2} \quad = \quad 171 \ .$$

EXAMPLE :

How many *nonnegative* integer solutions are there to the *inequality*

$$x_1 + x_2 + x_3 \leq 17 \ ?$$

SOLUTION :

Introduce an *auxiliary variable* (or "*slack variable*")

$$x_4 \equiv 17 - (x_1 + x_2 + x_3) \ .$$

Then

$$x_1 + x_2 + x_3 + x_4 = 17 \ .$$

Use seventeen 1's separated by 3 bars to indicate the possible values of x_1 , x_2 , x_3 , and x_4 , *e.g.*,

$$111|11111111|1111|11 \ .$$

$$111|11111111|1111|11 \ .$$

The total number of positions is

$$17 \ + \ 3 \ = \ 20 \ .$$

The total number of *nonnegative* solutions is therefore

$$\binom{20}{3} \quad = \quad \frac{20!}{(20-3)!\ 3!} \quad = \quad \frac{20 \times 19 \times 18}{3 \times 2} \quad = \quad 1140 \ .$$

EXAMPLE :

How many *positive* integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 17 \quad ?$$

SOLUTION : Let

$$x_1 = \tilde{x}_1 + 1 \quad , \quad x_2 = \tilde{x}_2 + 1 \quad , \quad x_3 = \tilde{x}_3 + 1 \ .$$

Then the problem becomes :

How many *nonnegative* integer solutions are there to the equation

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 14 \quad ?$$

$$111|111111111|11$$

The solution is

$$\binom{16}{2} = \frac{16!}{(16-2)!\ 2!} = \frac{16 \times 15}{2} = 120 \ .$$

EXAMPLE :

What is the probability the *sum* is 9 in *three rolls of a die* ?

SOLUTION : The number of such *sequences* of three rolls with sum 9 is the number of integer solutions of

$$x_1 + x_2 + x_3 = 9 ,$$

with

$$1 \le x_1 \le 6 , \quad 1 \le x_2 \le 6 , \quad 1 \le x_3 \le 6 .$$

Let

$$x_1 = \tilde{x}_1 + 1 , \quad x_2 = \tilde{x}_2 + 1 , \quad x_3 = \tilde{x}_3 + 1 .$$

Then the problem becomes :

How many *nonnegative* integer solutions are there to the equation

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 6 ,$$

with

$$0 \le \tilde{x}_1 , \tilde{x}_2 , \tilde{x}_3 \le 5 .$$

EXAMPLE : (continued $\cdots$)

Now the equation

$$\tilde{x}_1 \ + \ \tilde{x}_2 \ + \ \tilde{x}_3 \ = \ 6 \ , \qquad (\ 0 \ \le \ \tilde{x}_1 \ , \ \tilde{x}_2 \ , \ \tilde{x}_3 \ \le \ 5 \ ) \ ,$$

has

$$1|111|11$$

$$\binom{8}{2} \ = \ 28 \quad \text{solutions} \ ,$$

from which we must *subtract* the 3 *impossible* solutions

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \ = \ (6,0,0) \quad , \quad (0,6,0) \quad , \quad (0,0,6) \ .$$

$$111111|| \quad , \quad |111111| \quad , \quad ||111111$$

Thus the probability that the *sum* of 3 rolls equals 9 is

$$\frac{28 \ - \ 3}{6^3} \ = \ \frac{25}{216} \ \cong \ 0.116 \ .$$

EXAMPLE : (continued $\cdots$)

The 25 outcomes of the event "*the sum of the rolls is* 9" are

$$
\begin{array}{llllll}
\{ \ 126 \ , & 135 \ , & 144 \ , & 153 \ , & 162 \ , & \\
216 \ , & 225 \ , & 234 \ , & 243 \ , & 252 \ , & 261 \ , \\
315 & 324 \ , & 333 \ , & 342 \ , & 351 \ , & \\
414 \ , & 423 \ , & 432 \ , & 441 \ , & & \\
513 \ , & 522 \ , & 531 \ , & & & \\
612 \ , & 621 & \} \ . & & &
\end{array}
$$

The "lexicographic" ordering of the *outcomes* (which are *sequences*) in this *event* is used for systematic counting.

EXERCISE :

- How many integer solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 17 ,$$

if we require that

$$x_1 \geq 1 , \quad x_2 \geq 2 , \quad x_3 \geq 3 ?$$

EXERCISE :

What is the probability that the *sum* is *less than or equal to* 9 in *three rolls of a die* ?

CONDITIONAL PROBABILITY

Giving more information can change the probability of an event.

EXAMPLE :

If a coin is tossed two times then what is the probability of two Heads?

ANSWER :

$$\frac{1}{4} .$$

EXAMPLE :

If a coin is tossed two times then what is the probability of two Heads, *given that the first toss gave Heads* ?

ANSWER :

$$\frac{1}{2} .$$

NOTE :

Several examples will be about *playing cards* .

A standard *deck* of *playing cards* consists of 52 cards :

- Four *suits* :

  Hearts , Diamonds (*red*) , and Spades , Clubs (*black*) .

- Each suit has 13 cards, whose *denomination* is

  $2\ ,\ 3\ ,\ \cdots\ ,10$, Jack , Queen , King , Ace .

- The Jack , Queen , and King are called *face cards* .

EXERCISE :

Suppose we draw a card from a shuffled set of 52 playing cards.

- What is the probability of drawing a Queen ?

- What is the probability of drawing a Queen, given that the card drawn is of *suit* Hearts ?

- What is the probability of drawing a Queen, given that the card drawn is a *Face card* ?

What do the answers tell us?

(We'll soon learn the events "Queen" and "Hearts" are *independent* .)

The two preceding questions are examples of *conditional probability* .

Conditional probability is an *important* and *useful* concept.

If E and F are events, *i.e.*, subsets of a sample space $\mathcal{S}$, then

$P(E|F)$ *is the conditional probability of* E *, given* F ,

defined as

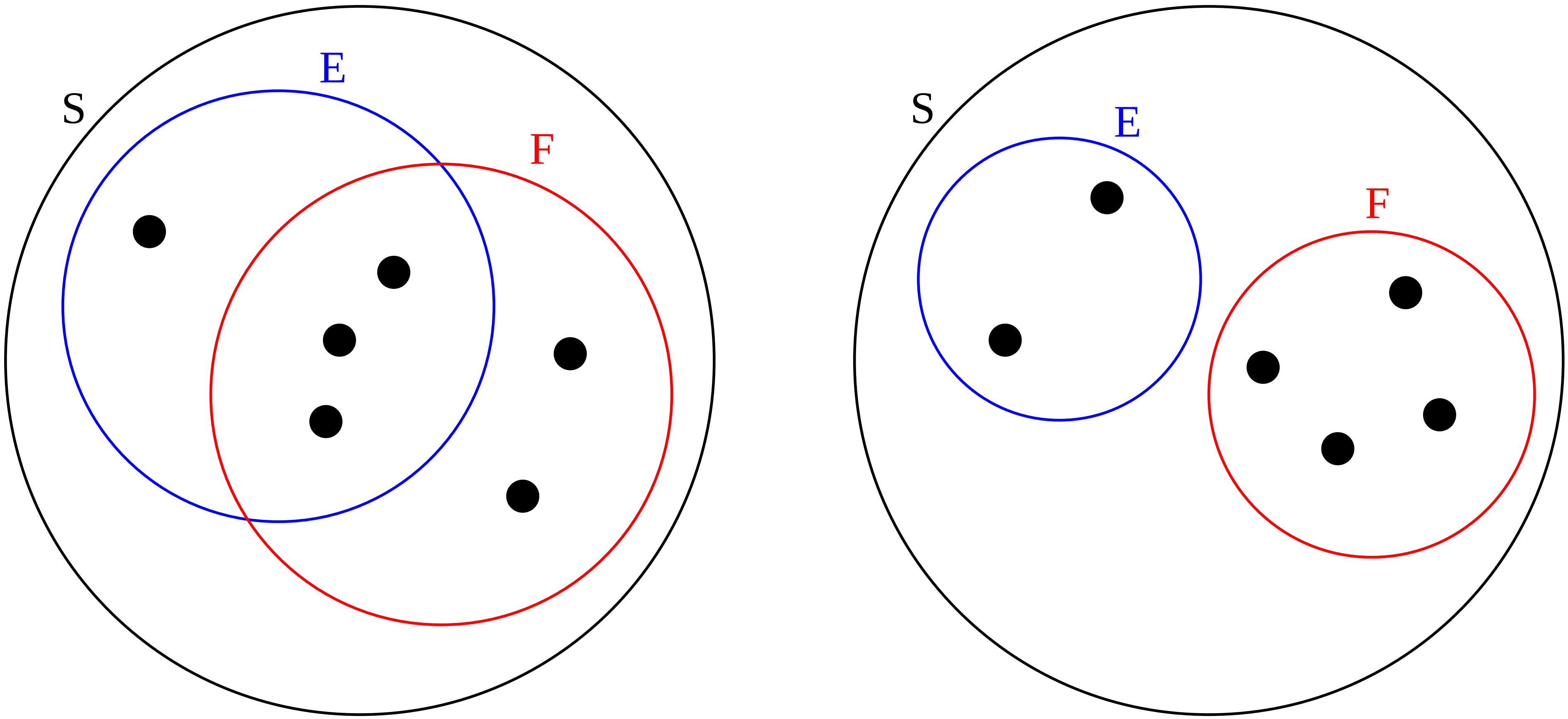
$$P(E|F) \equiv \frac{P(EF)}{P(F)} .$$

or, equivalently

$$P(EF) = P(E|F)\, P(F) ,$$

(assuming that $P(F)$ is not zero).

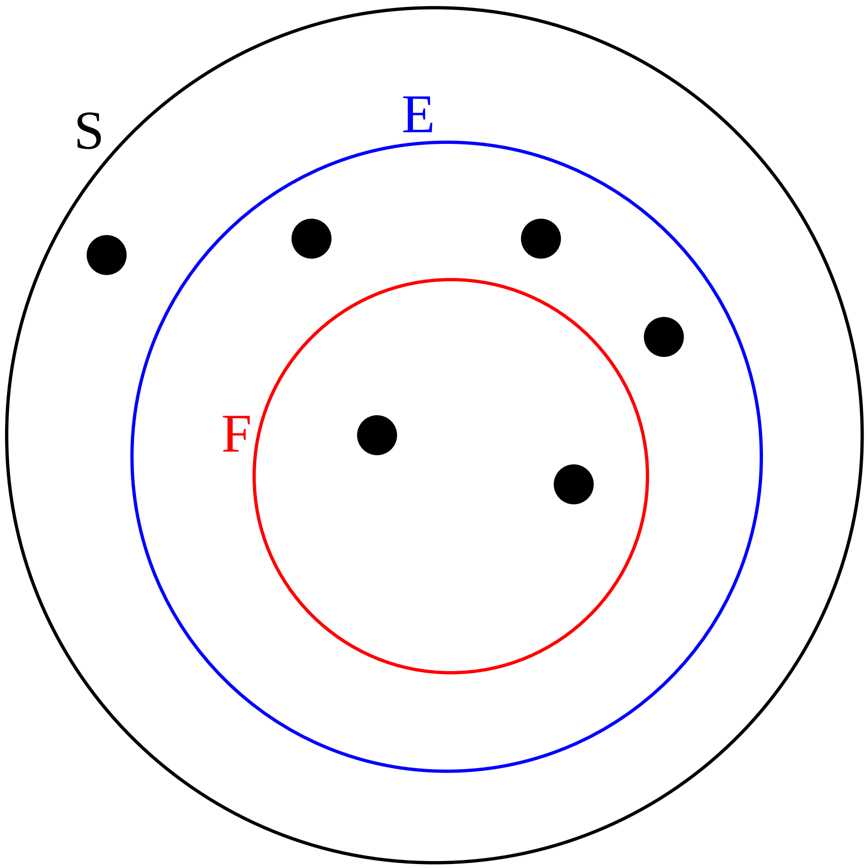
$$P(E|F) \equiv \frac{P(EF)}{P(F)}$$

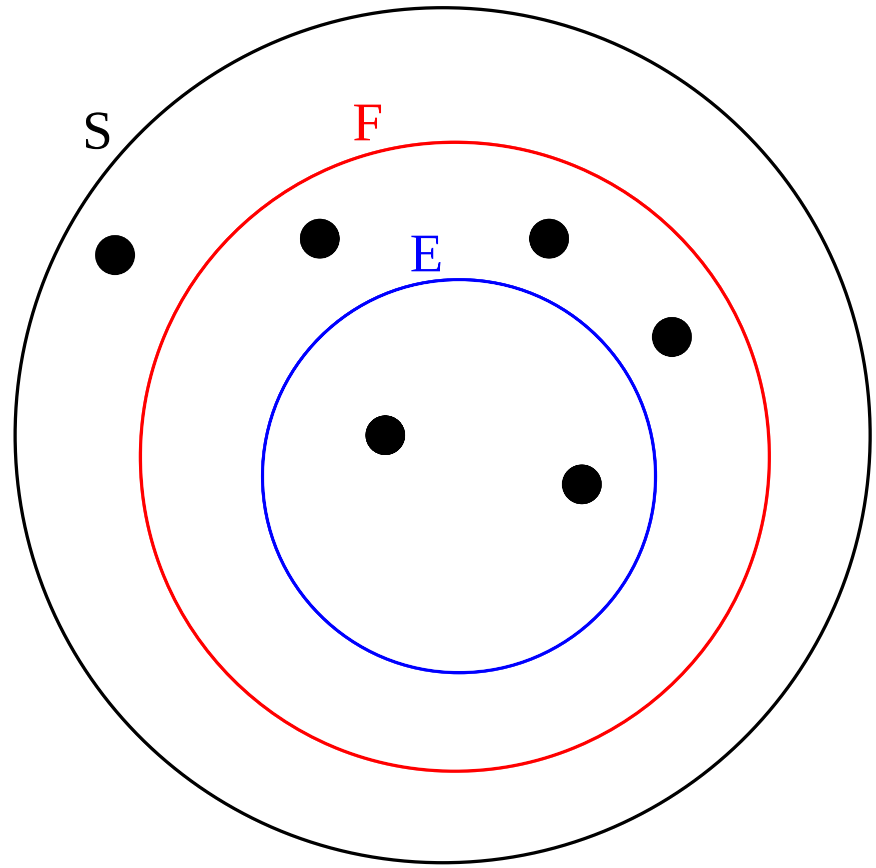


Suppose that the 6 outcomes in $\mathcal{S}$ are equally likely.

What is $P(E|F)$ in each of these two cases ?

$$P(E|F) \equiv \frac{P(EF)}{P(F)}$$





Suppose that the 6 outcomes in $\mathcal{S}$ are equally likely.

What is $P(E|F)$ in each of these two cases ?

EXAMPLE : Suppose a coin is tossed two times.

The sample space is

$$\mathcal{S} \ = \ \{HH \ , \ HT \ , \ TH \ , \ TT\} \ .$$

Let E be the event "*two Heads*" , *i.e.*,

$$E \ = \ \{HH\} \ .$$

Let F be the event "*the first toss gives Heads*" , *i.e.*,

$$F \ = \ \{HH \ , \ HT\} \ .$$

Then

$$EF \ = \ \{HH\} \ = \ E \qquad (\text{ since } \ E \subset F \ ) \ .$$

We have

$$P(E|F) \ = \ \frac{P(EF)}{P(F)} \ = \ \frac{P(E)}{P(F)} \ = \ \frac{\frac{1}{4}}{\frac{2}{4}} \ = \ \frac{1}{2} \ .$$

EXAMPLE :

Suppose we draw a card from a shuffled set of 52 playing cards.

- What is the probability of drawing a Queen, given that the card drawn is of *suit* Hearts ?

  ANSWER :

$$P(Q|H) = \frac{P(QH)}{P(H)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13} .$$

- What is the probability of drawing a Queen, given that the card drawn is a *Face card* ?

  ANSWER :

$$P(Q|F) = \frac{P(QF)}{P(F)} = \frac{P(Q)}{P(F)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3} .$$

(Here $Q \subset F$, so that $QF = Q$.)

The probability of an event E is sometimes computed more easily

if we condition E on another event F ,

namely, from

$$P(E) \quad = \quad P(\ E(F \cup F^c)\ ) \qquad (\text{ Why ? })$$

$$= \quad P(\ EF \cup EF^c\ ) \quad = \quad P(EF)\ +\ P(EF^c) \qquad (\text{ Why ? })$$

and

$$P(EF) \ = \ P(E|F)\ P(F) \quad , \quad P(EF^c) \ = \ P(E|F^c)\ P(F^c)\ ,$$

we obtain this *basic formula*

$$P(E) \quad = \quad P(E|F) \cdot P(F) \quad + \quad P(E|F^c) \cdot P(F^c)\ .$$

EXAMPLE :

An insurance company has these data :

The probability of an insurance claim in a period of one year is

4 percent for persons under age 30

2 percent for persons over age 30

and it is known that

30 percent of the targeted population is under age 30.

What is the probability of an insurance claim in a period of one year for a randomly chosen person from the targeted population?

SOLUTION :

Let the sample space $\mathcal{S}$ be all persons under consideration.

Let C be the event (subset of $\mathcal{S}$) of persons filing a claim.

Let U be the event (subset of $\mathcal{S}$) of persons under age 30.

Then U^c is the event (subset of $\mathcal{S}$) of persons over age 30.

Thus

$$\begin{aligned}
P(C) &= P(C|U)\ P(U) \ + \ P(C|U^c)\ P(U^c) \\
&= \frac{4}{100}\ \frac{3}{10} + \frac{2}{100}\ \frac{7}{10} \\
&= \frac{26}{1000} \quad = \quad 2.6\% \ .
\end{aligned}$$

EXAMPLE :

Two balls are drawn from a bag with 2 *white* and 3 *black* balls.

There are 20 outcomes (*sequences*) in $\mathcal{S}$. (Why ?)

What is the probability that *the second ball is white* ?

SOLUTION :

Let F be the event that *the first ball is white*.

Let S be the event that *the second second ball is white*.

Then

$$P(S) \; = \; P(S|F)\,P(F) + P(S|F^c)\,P(F^c) \; = \; \frac{1}{4}\cdot\frac{2}{5} \; + \; \frac{2}{4}\cdot\frac{3}{5} \; = \; \frac{2}{5}.$$

QUESTION : Is it surprising that $P(S) \; = \; P(F)$?

EXAMPLE : (continued $\cdots$)

Is it surprising that $P(S) \; = \; P(F)$?

ANSWER : Not really, if one considers the sample space $\mathcal{S}$:

$$\begin{array}{lllll} \{ & \mathbf{w}_1\mathbf{w}_2 \, , & \mathbf{w}_1 b_1 \, , & \mathbf{w}_1 b_2 \, , & \mathbf{w}_1 b_3 \, , \\ & \mathbf{w}_2\mathbf{w}_1 \, , & \mathbf{w}_2 b_1 \, , & \mathbf{w}_2 b_2 \, , & \mathbf{w}_2 b_3 \, , \\ & b_1\mathbf{w}_1 \, , & b_1\mathbf{w}_2 \, , & b_1 b_2 \, , & b_1 b_3 \, , \\ & b_2\mathbf{w}_1 \, , & b_2\mathbf{w}_2 \, , & b_2 b_1 \, , & b_2 b_3 \, , \\ & b_3\mathbf{w}_1 \, , & b_3\mathbf{w}_2 \, , & b_3 b_1 \, , & b_3 b_2 \quad \} \, , \end{array}$$

where outcomes (*sequences*) are assumed equally likely.

EXAMPLE :

Suppose we draw *two cards* from a shuffled set of 52 playing cards.

What is the probability that the second card is a Queen ?

ANSWER :

$$P(2^{\text{nd}} \text{ card } Q) \quad =$$

$$P(2^{\text{nd}} \text{ card } Q | 1^{\text{st}} \text{ card } Q) \quad \cdot \quad P(1^{\text{st}} \text{ card } Q)$$

$$+ \quad P(2^{\text{nd}} \text{ card } Q | 1^{\text{st}} \text{ card not } Q) \quad \cdot \quad P(1^{\text{st}} \text{ card not } Q)$$

$$= \quad \frac{3}{51} \cdot \frac{4}{52} \quad + \quad \frac{4}{51} \cdot \frac{48}{52} \quad = \quad \frac{204}{51 \cdot 52} \quad = \quad \frac{4}{52} \quad = \quad \frac{1}{13} \ .$$

QUESTION : Is it surprising that $P(2^{\text{nd}} \text{ card } Q) \ = \ P(1^{\text{st}} \text{ card } Q)$?

A useful formula that *"inverts conditioning"* is derived as follows :

Since we have both

$$P(EF) \quad = \quad P(E|F)\ P(F)\ ,$$

and

$$P(EF) \quad = \quad P(F|E)\ P(E)\ .$$

If $P(E) \neq 0$ then it follows that

$$P(F|E) \quad = \quad \frac{P(EF)}{P(E)} \quad = \quad \frac{P(E|F) \cdot P(F)}{P(E)}\ ,$$

and, using the earlier useful formula, we get

$$P(F|E) \quad = \quad \frac{P(E|F) \cdot P(F)}{P(E|F) \cdot P(F)\ +\ P(E|F^c) \cdot P(F^c)}\ ,$$

which is known as *Bayes' formula* .

EXAMPLE : Suppose 1 in 1000 persons has a certain disease.

A test detects the disease in 99 % of diseased persons.

The test also "detects" the disease in 5 % of healthly persons.

With what probability does a positive test diagnose the disease?

SOLUTION : Let

$$D \sim \text{"diseased"} \quad , \quad H \sim \text{"healthy"} \quad , \quad + \sim \text{"positive"} .$$

We are given that

$$P(D) = 0.001 \, , \qquad P(+|D) = 0.99 \, , \qquad P(+|H) = 0.05 \, .$$

By Bayes' formula

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+|D) \cdot P(D) + P(+|H) \cdot P(H)}$$

$$= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 \; + \; 0.05 \cdot 0.999} \cong 0.0194 \qquad (\,!\,)$$

EXERCISE :

Suppose 1 in 100 products has a certain defect.

A test detects the defect in 95 % of defective products.

The test also "detects" the defect in 10 % of non-defective products.

- With what probability does a positive test diagnose a defect?

EXERCISE :

Suppose 1 in 2000 persons has a certain disease.

A test detects the disease in 90 % of diseased persons.

The test also "detects" the disease in 5 % of healthly persons.

- With what probability does a positive test diagnose the disease?

More generally, if the sample space $\mathcal{S}$ is *the union of disjoint events*

$$\mathcal{S} \ = \ F_1 \cup F_2 \cup \cdots \cup F_n \ ,$$

then for any event E

$$P(F_i|E) \ = \ \frac{P(E|F_i) \cdot P(F_i)}{P(E|F_1) \cdot P(F_1) \ + \ P(E|F_2) \cdot P(F_2) \ + \cdots + \ P(E|F_n) \cdot P(F_n)}$$

EXERCISE :

Machines M_1, M_2, M_3 produce these *proportions* of a article

Production : $M_1 \ : \ 10\ \%$, $M_2 \ : \ 30\ \%$, $M_3 \ : \ 60\ \%$.

The probability the machines produce *defective* articles is

Defects : $M_1 \ : \ 4\ \%$, $M_2 \ : \ 3\ \%$, $M_3 \ : \ 2\ \%$.

What is the probability a random article was made by machine M_1, given that it is defective?

Independent Events

Two events E and F are *independent* if

$$P(EF) \ = \ P(E)\ P(F) \ .$$

In this case

$$P(E|F) \ = \ \frac{P(EF)}{P(F)} \ = \ \frac{P(E)\ P(F)}{P(F)} \ = \ P(E) \ ,$$

(assuming $P(F)$ is not zero).

Thus

knowing F *occurred doesn't change the probability of* E *.*

EXAMPLE : Draw *one* card from a deck of 52 playing cards.

Counting outcomes we find

$$
\begin{aligned}
P(\text{Face Card}) &= \tfrac{12}{52} = \tfrac{3}{13}\,, \\
P(\text{Hearts}) &= \tfrac{13}{52} = \tfrac{1}{4}\,, \\
P(\text{Face Card and Hearts}) &= \tfrac{3}{52}\,, \\
P(\text{Face Card}|\text{Hearts}) &= \tfrac{3}{13}\,.
\end{aligned}
$$

We see that

$$P(\text{Face Card and Hearts}) \;=\; P(\text{Face Card}) \cdot P(\text{Hearts}) \quad (= \frac{3}{52}\,)\,.$$

Thus the events "*Face Card*" and "*Hearts*" are *independent*.

Therefore we also have

$$P(\text{Face Card}|\text{Hearts}) \;=\; P(\text{Face Card}) \quad (= \frac{3}{13}\,)\,.$$

EXERCISE :

Which of the following pairs of events are independent?

(1) drawing "Hearts" and drawing "Black" ,

(2) drawing "Black" and drawing "Ace" ,

(3) the event $\{2, 3, \cdots, 9\}$ and drawing "Red" .

EXERCISE : *Two* numbers are drawn at random from the set

$$\{ 1 , 2 , 3 , 4 \} .$$

If *order is not important* then what is the sample space $\mathcal{S}$?

Define the following functions on $\mathcal{S}$:

$$X(\{i,j\}) \quad = \quad i + j , \qquad\qquad Y(\{i,j\}) \quad = \quad |i - j| .$$

Which of the following pairs of events are independent?

$$(1) \quad X = 5 \quad \text{and} \quad Y = 2 ,$$

$$(2) \quad X = 5 \quad \text{and} \quad Y = 1 .$$

REMARK :

X and Y are examples of *random variables* . (More soon!)

EXAMPLE : If E and F are *independent* then so are E and F^c .

PROOF : $E = E(F \cup F^c) = EF \cup EF^c$, where

EF and EF^c are *disjoint* .

Thus

$$P(E) = P(EF) + P(EF^c) ,$$

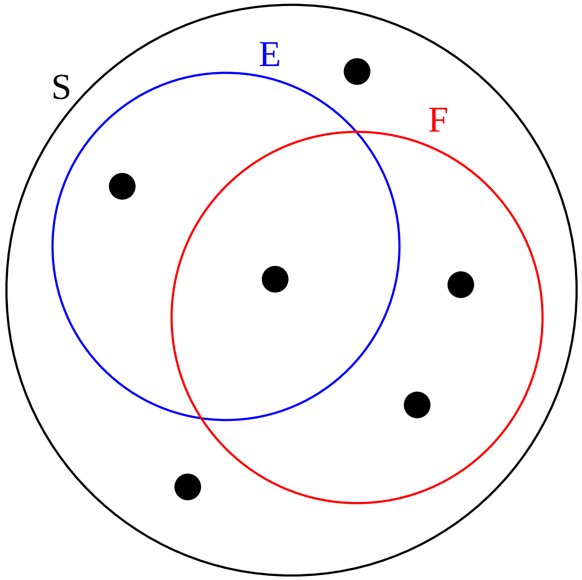
from which

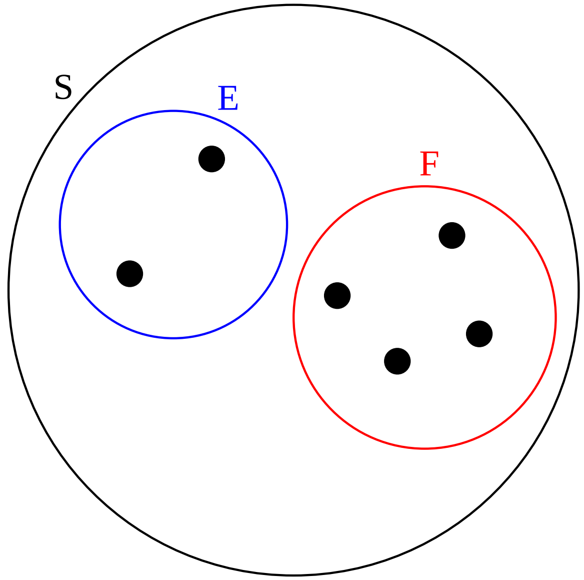
$$\begin{aligned} P(EF^c) &= P(E) - P(EF) \\ &= P(E) - P(E) \cdot P(F) \quad \text{(since } E \text{ and } F \text{ independent)} \\ &= P(E) \cdot (1 - P(F)) \\ &= P(E) \cdot P(F^c) . \end{aligned}$$

EXERCISE :

Prove that if E and F are *independent* then so are E^c and F^c .

NOTE : *Independence* and *disjointness* *are different things* !





Independent, but not disjoint. Disjoint, but not independent.

(The six outcomes in S are assumed to have equal probability.)

If E and F are *independent* then $P(EF) = P(E)\ P(F)$.

If E and F are *disjoint* then $P(EF) = P(\ \emptyset\ ) = 0$.

If E and F are *independent and disjoint* then one has *zero probability* !

Three events E , F, and G are *independent* if

$$P(EFG) \quad = \quad P(E)\ P(F)\ P(G)\ .$$

and

$$\begin{aligned} P(EF) &= P(E)\ P(F)\ . \\ P(EG) &= P(E)\ P(G)\ . \\ P(FG) &= P(F)\ P(G)\ . \end{aligned}$$

EXERCISE : Are the three events of drawing

(1) a red card ,

(2) a face card ,

(3) a Heart or Spade ,

independent ?

EXERCISE :

A machine M consists of three *independent parts*, M_1, M_2, and M_3 .

Suppose that

$$M_1 \quad \text{functions properly with probability} \quad \tfrac{9}{10} \ ,$$

$$M_2 \quad \text{functions properly with probability} \quad \tfrac{9}{10} \ ,$$

$$M_3 \quad \text{functions properly with probability} \quad \tfrac{8}{10} \ ,$$

and that

the machine M functions if and only if *its three parts function*.

- What is the probability for the machine M to *function* ?
- What is the probability for the machine M to *malfunction* ?

DISCRETE RANDOM VARIABLES

DEFINITION : A *discrete random variable* is a *function* $X(s)$ from a *finite* or *countably infinite* sample space $\mathcal{S}$ to the real numbers :

$$X(\cdot) \quad : \quad \mathcal{S} \quad \rightarrow \quad \mathbb{R} \ .$$

EXAMPLE : Toss a coin 3 times in sequence. The sample space is

$$\mathcal{S} \ = \ \{HHH \, , \, HHT \, , \, HTH \, , \, HTT \, , \, THH \, , \, THT \, , \, TTH \, , \, TTT\} \, ,$$

and examples of random variables are

- $X(s) \ = \$ the number of Heads in the sequence ; *e.g.*, $X(HTH) \ = \ 2$,

- $Y(s) \ = \$ The index of the first H ; *e.g.*, $Y(TTH) \ = \ 3$,

  0 if the sequence has no H , *i.e.*, $Y(TTT) \ = \ 0$.

NOTE : In this example $X(s)$ and $Y(s)$ are actually *integers* .

Value-ranges of a random variable correspond to *events* in $\mathcal{S}$.

EXAMPLE : For the sample space

$\mathcal{S} \;=\; \{HHH\,,\,HHT\,,\,HTH\,,\,HTT\,,\,THH\,,\,THT\,,\,TTH\,,\,TTT\}$,

with

$$X(s) \;=\; \text{the number of Heads}\ ,$$

the value

$X(s) \;=\; 2$,   corresponds to the event $\{HHT\ ,\ HTH\ ,\ THH\}$,

and the values

$1 < X(s) \le 3$,   correspond to $\{HHH\ ,\ HHT\ ,\ HTH\ ,\ THH\}$.

NOTATION : If it is clear what $\mathcal{S}$ is then we often just write

$$X \qquad \text{instead of} \qquad X(s)\ .$$

Value-ranges of a random variable correspond to *events* in $\mathcal{S}$,

and

$$\textit{events} \text{ in } \mathcal{S} \text{ have a } \textit{probability} \; .$$

Thus

$$\textit{Value-ranges} \text{ of a random variable have a } \textit{probability} \; .$$

EXAMPLE : For the sample space

$$\mathcal{S} \;=\; \{HHH\,,\, HHT\,,\, HTH\,,\, HTT\,,\, THH\,,\, THT\,,\, TTH\,,\, TTT\}\,,$$

with

$$X(s) \;=\; \text{the number of Heads}\;,$$

we have

$$P(0 < X \le 2) \;=\; \frac{6}{8}\;.$$

QUESTION : What are the values of

$P(X \le -1)\,,\ P(X \le 0)\,,\ P(X \le 1)\,,\ P(X \le 2)\,,\ P(X \le 3)\,,\ P(X \le 4)\,?$

NOTATION : We will also write $p_X(x)$ to denote $P(X = x)$.

EXAMPLE : For the sample space

$$\mathcal{S} = \{HHH\,,\, HHT\,,\, HTH\,,\, HTT\,,\, THH\,,\, THT\,,\, TTH\,,\, TTT\}\,,$$

with

$$X(s) = \text{the number of Heads}\,,$$

we have

$$p_X(0) \equiv P(\,\{TTT\}\,) = \tfrac{1}{8}$$

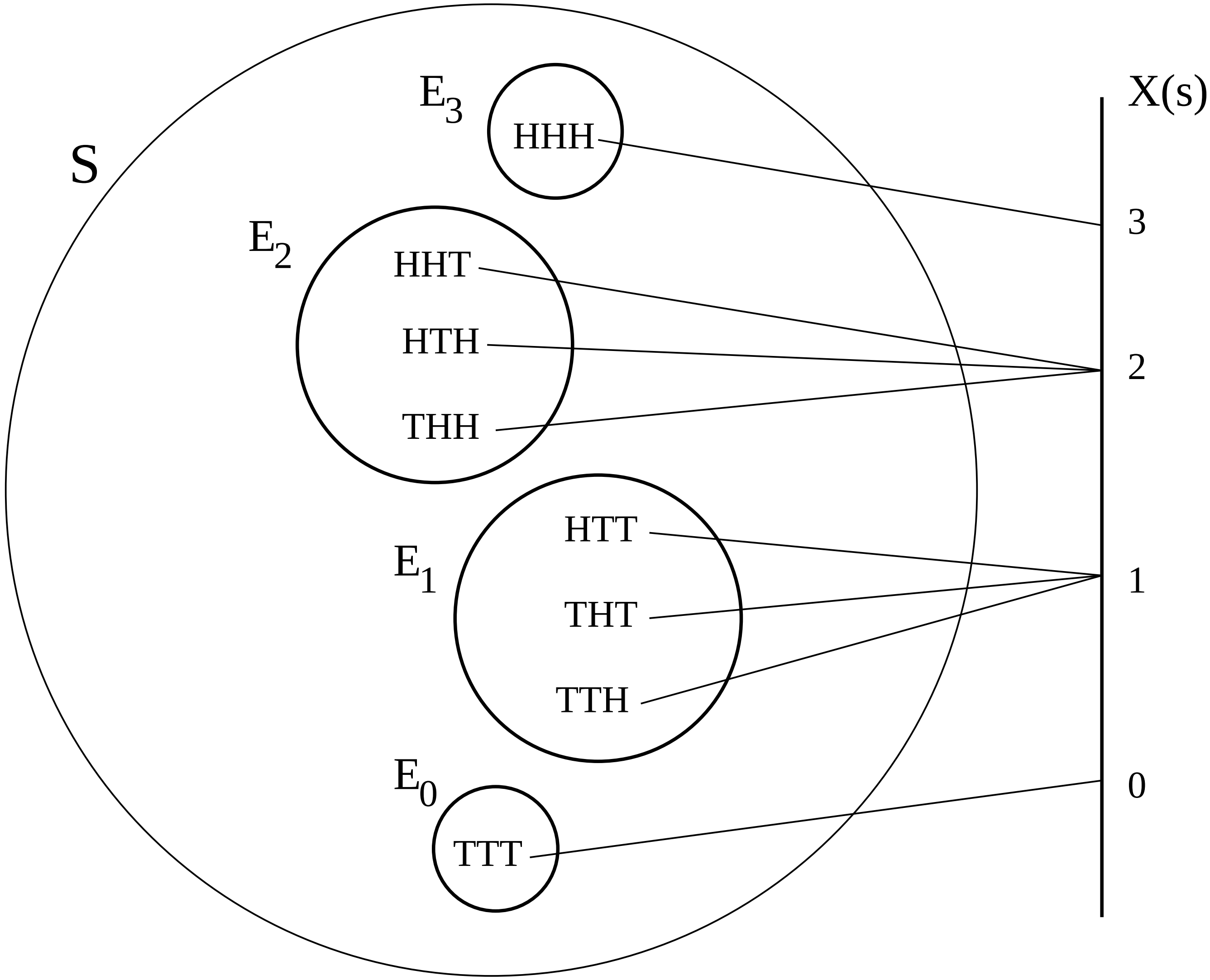
$$p_X(1) \equiv P(\,\{HTT\,,\, THT\,,\, TTH\}\,) = \tfrac{3}{8}$$

$$p_X(2) \equiv P(\,\{HHT\,,\, HTH\,,\, THH\}\,) = \tfrac{3}{8}$$

$$p_X(3) \equiv P(\,\{HHH\}\,) = \tfrac{1}{8}$$

where

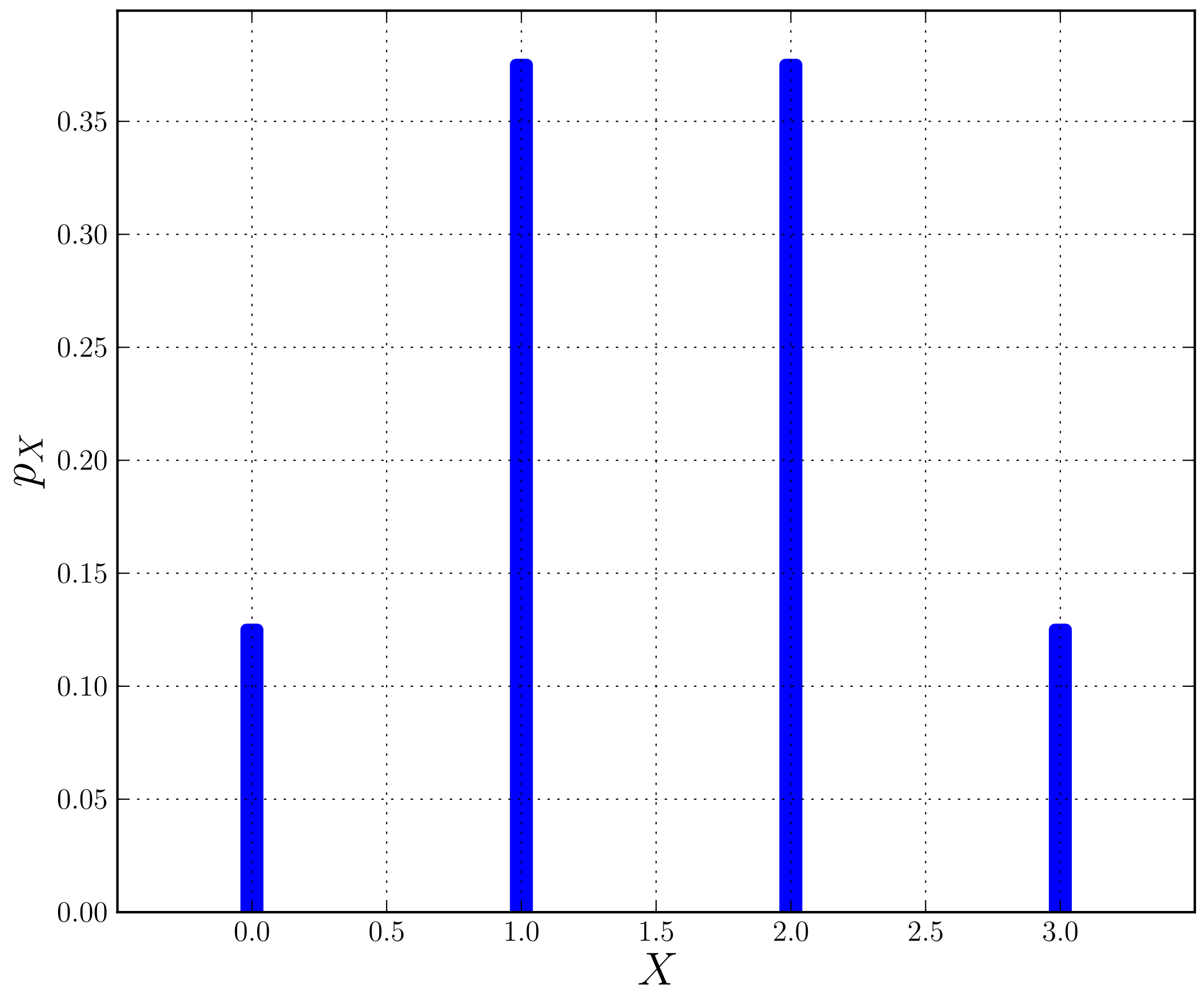
$$p_X(0) + p_X(1) + p_X(2) + p_X(3) = 1\,. \qquad (\text{ Why ? })$$



Graphical representation of X .

The *events* E_0, E_1, E_2, E_3 are *disjoint* since $X(s)$ is a *function* !

$(X : S \rightarrow \mathbb{R}$ must be defined for *all* $s \in S$ and must be *single-valued*.)



The graph of p_X .

DEFINITION :
$$p_X(x) \equiv P(X = x) ,$$
is called the *probability mass function* .

DEFINITION :
$$F_X(x) \equiv P(X \le x) ,$$
is called the (*cumulative*) *probability distribution function* .

PROPERTIES :

- $F_X(x)$ is a *non-decreasing* function of x . (Why ?)
- $F_X(-\infty) = 0$ and $F_X(\infty) = 1$. (Why ?)
- $P(a < X \le b) = F_X(b) - F_X(a)$. (Why ?)

NOTATION : When it is clear what X is then we also write

$$p(x) \text{ for } p_X(x) \quad \text{and} \quad F(x) \text{ for } F_X(x) .$$

EXAMPLE : With $X(s) =$ the number of Heads , and

$\mathcal{S} = \{HHH\,,\, HHT\,,\, HTH\,,\, HTT\,,\, THH\,,\, THT\,,\, TTH\,,\, TTT\}$,

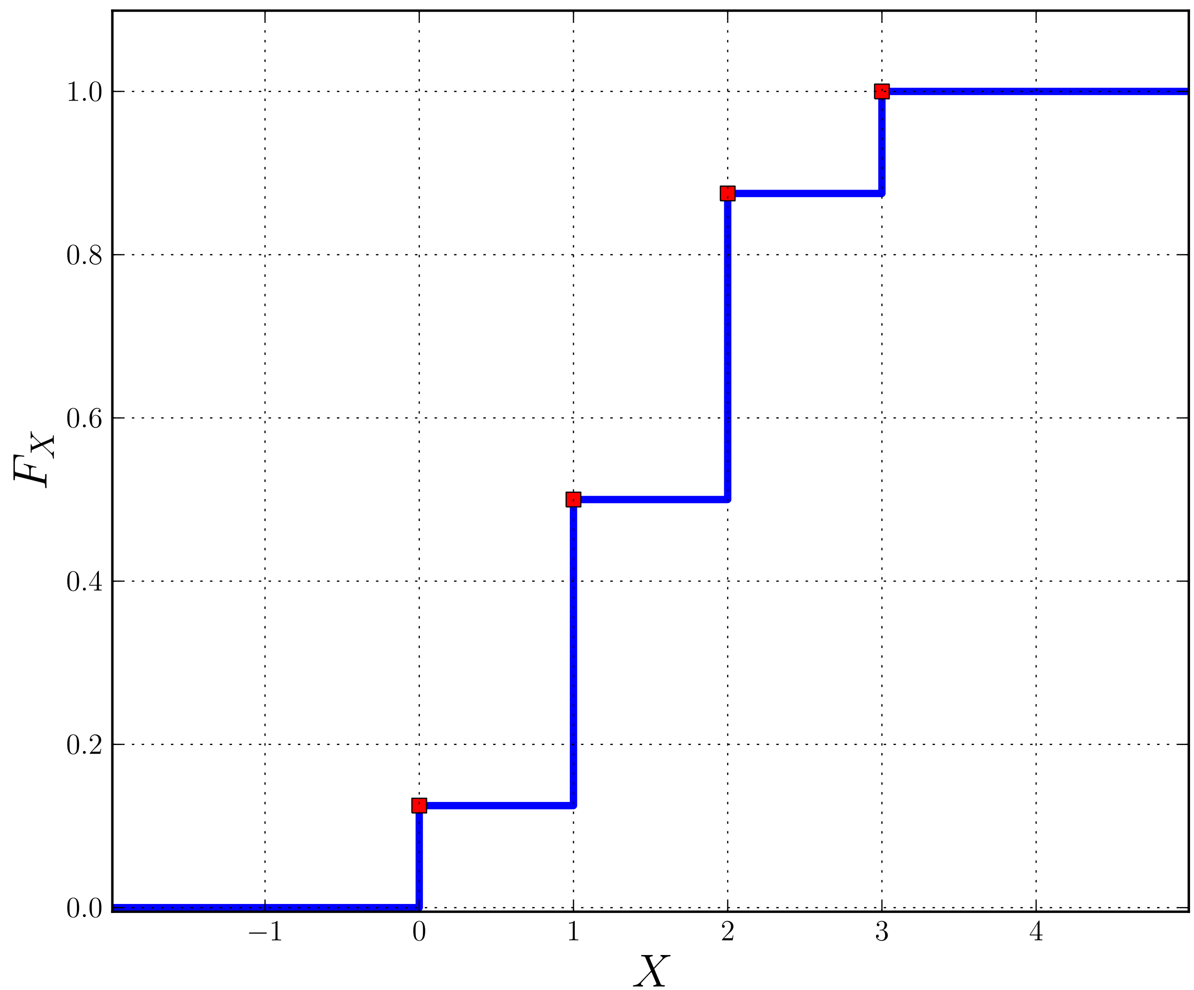
$$p(0) = \tfrac{1}{8} \quad , \quad p(1) = \tfrac{3}{8} \quad , \quad p(2) = \tfrac{3}{8} \quad , \quad p(3) = \tfrac{1}{8} \ ,$$

we have the *probability distribution function*

$$\begin{array}{lclcl}
F(-1) & \equiv & P(X \leq -1) & = & 0 \\
F(\ 0) & \equiv & P(X \leq \ 0) & = & \mathbf{\frac{1}{8}} \\
F(\ 1) & \equiv & P(X \leq \ 1) & = & \frac{4}{8} \\
F(\ 2) & \equiv & P(X \leq \ 2) & = & \mathbf{\frac{7}{8}} \\
F(\ 3) & \equiv & P(X \leq \ 3) & = & 1 \\
F(\ 4) & \equiv & P(X \leq \ 4) & = & 1
\end{array}$$

We see, for example, that

$$\begin{aligned}
P(0 < X \leq 2) &= P(X = 1) + P(X = 2) \\
&= F(2) - F(0) = \tfrac{7}{8} - \tfrac{1}{8} = \tfrac{6}{8} \ .
\end{aligned}$$



The graph of the *probability distribution function* F_X .

EXAMPLE : Toss a coin until "Heads" occurs.

Then the sample space is *countably infinite* , namely,

$$\mathcal{S} \;=\; \{H \;,\; TH \;,\; TTH \;,\; TTTH \;,\; \cdots \} \,.$$

The *random variable* X is the *number of rolls* until "Heads" occurs :

$$X(H) = 1 \quad , \quad X(TH) = 2 \quad , \quad X(TTH) = 3 \quad , \quad \cdots$$

Then

$$p(1) = \tfrac{1}{2} \quad , \quad p(2) = \tfrac{1}{4} \quad , \quad p(3) = \tfrac{1}{8} \quad , \quad \cdots \qquad (\text{ Why ? })$$

and

$$F(n) \;=\; P(X \le n) \;=\; \sum_{k=1}^{n} p(k) \;=\; \sum_{k=1}^{n} \frac{1}{2^k} \;=\; 1 - \frac{1}{2^n} \,,$$

and, as should be the case,

$$\sum_{k=1}^{\infty} p(k) \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} p(k) \;=\; \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) \;=\; 1 \,.$$

NOTE : The outcomes in $\mathcal{S}$ *do not have equal probability* !

EXERCISE : Draw the *probability mass* and *distribution functions*.

$X(s)$ is the *number of tosses* until "Heads" occurs $\cdots$

REMARK : We can also take $\mathcal{S} \equiv \mathcal{S}_n$ as *all ordered outcomes of length* n. For example, for $n = 4$,

$$\mathcal{S}_4 = \{ \tilde{H}HHH, \tilde{H}HHT, \tilde{H}HTH, \tilde{H}HTT,$$
$$\tilde{H}THH, \tilde{H}THT, \tilde{H}TTH, \tilde{H}TTT,$$
$$T\tilde{H}HH, T\tilde{H}HT, T\tilde{H}TH, T\tilde{H}TT,$$
$$TT\tilde{H}H, TT\tilde{H}T, TTT\tilde{H}, TTTT \} .$$

where for each outcome the first "Heads" is marked as $\tilde{H}$.

Each outcome in $\mathcal{S}_4$ has *equal probability* 2^{-n} (here $2^{-4} = \frac{1}{16}$) , and

$$p_X(1) = \tfrac{1}{2} \quad , \quad p_X(2) = \tfrac{1}{4} \quad , \quad p_X(3) = \tfrac{1}{8} \quad , \quad p_X(4) = \tfrac{1}{16} \quad \cdots ,$$

independent of n .

Joint distributions

The *probability mass function* and the *probability distribution function* can also be functions of *more than one variable*.

EXAMPLE : Toss a coin 3 times in sequence. For the sample space

$$\mathcal{S} \;=\; \{HHH\,,\, HHT\,,\, HTH\,,\, HTT\,,\, THH\,,\, THT\,,\, TTH\,,\, TTT\}\,,$$

we let

$$X(s) \;=\; \#\ \text{Heads} \quad , \quad Y(s) \;=\; \text{index of the first } H \quad (0 \text{ for } TTT)\,.$$

Then we have the *joint probability mass function*

$$p_{X,Y}(x,y) \;=\; P(X=x\;,\;Y=y)\,.$$

For example,

$$\begin{aligned} p_{X,Y}(2,1) \;&=\; P(X=2\;,\;Y=1) \\ &=\; P(\;2\ \text{Heads}\quad,\;1^{\text{st}}\ \text{toss is Heads}) \\ &=\; \tfrac{2}{8} \;=\; \tfrac{1}{4}\,. \end{aligned}$$

EXAMPLE : (continued $\cdots$) For

$\mathcal{S} = \{HHH\,,\, HHT\,,\, HTH\,,\, HTT\,,\, THH\,,\, THT\,,\, TTH\,,\, TTT\}$,

$X(s) =$ number of Heads, and $Y(s) =$ index of the first H ,

we can list the values of $p_{X,Y}(x,y)$:

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$\mathbf{y=1}$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$\mathbf{x=2}$	0	$\mathbf{\frac{2}{8}}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

NOTE :

- The *marginal probability* p_X is the probability mass function of X.
- The *marginal probability* p_Y is the probability mass function of Y.

EXAMPLE : (continued $\cdots$)

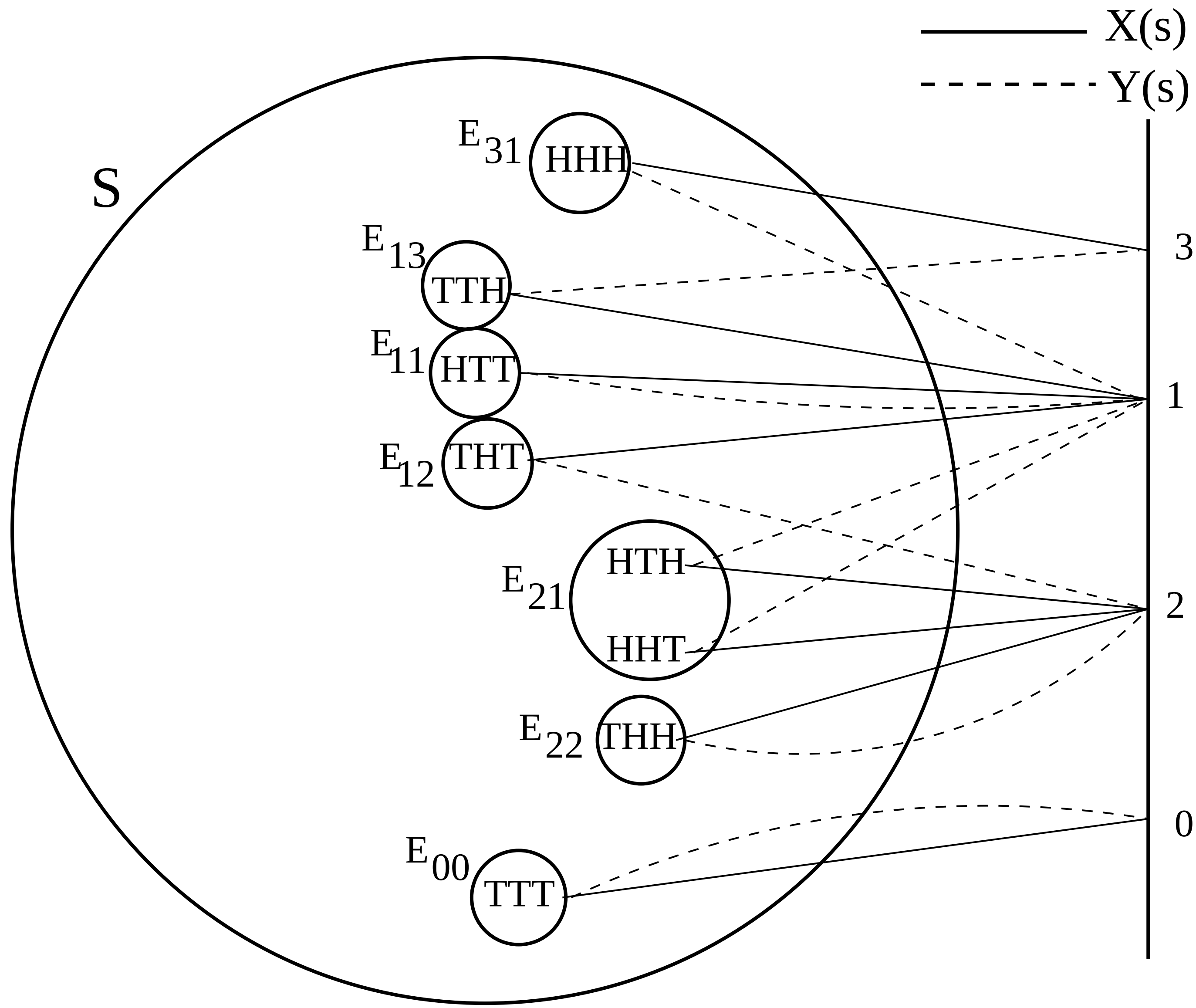
$X(s)$ = number of Heads, and $Y(s)$ = index of the first H .

	$y = 0$	$\mathbf{y = 1}$	$y = 2$	$y = 3$	$p_X(x)$
$x = 0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x = 1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$\mathbf{x = 2}$	0	$\mathbf{\frac{2}{8}}$	$\frac{1}{8}$	0	$\mathbf{\frac{3}{8}}$
$x = 3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\mathbf{\frac{4}{8}}$	$\frac{2}{8}$	$\frac{1}{8}$	1

For example,

- $X = 2$ corresponds to the *event* $\{HHT\ ,\ HTH\ ,\ THH\}$.
- $Y = 1$ corresponds to the *event* $\{HHH\ ,\ HHT\ ,\ HTH\ ,\ HTT\}$.
- $(X = 2$ *and* $Y = 1)$ corresponds to the *event* $\{HHT\ ,\ HTH\}$.

QUESTION : Are the events $X = 2$ and $Y = 1$ *independent* ?



The *events* $E_{i,j} \equiv \{ \ s \in S \ : \ X(s) = i \ , \ Y(s) = j \ \}$ are *disjoint* .

QUESTION : Are the events $X = 2$ and $Y = 1$ *independent* ?

DEFINITION :

$$p_{X,Y}(x,y) \quad \equiv \quad P(X = x \ , \ Y = y) \ ,$$

is called the *joint probability mass function* .

DEFINITION :

$$F_{X,Y}(x,y) \quad \equiv \quad P(X \leq x \ , \ Y \leq y) \ ,$$

is called the *joint* (*cumulative*) *probability distribution function* .

NOTATION : When it is clear what X and Y are then we also write

$$p(x,y) \quad \text{for} \quad p_{X,Y}(x,y) \ ,$$

and

$$F(x,y) \quad \text{for} \quad F_{X,Y}(x,y) \ .$$

EXAMPLE : *Three* tosses : $X(s) = \#$ Heads, $Y(s) =$ index 1^{st} H .

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

Joint distribution function $F_{X,Y}(x,y) \equiv P(X \leq x, Y \leq y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$F_X(\cdot)$
$x=0$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$x=1$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{4}{8}$
$x=2$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{7}{8}$
$x=3$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	1	1
$F_Y(\cdot)$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	1	1

Note that the distribution function F_X is a *copy* of the 4th column, and the distribution function F_Y is a *copy* of the 4th row. (Why ?)

In the preceding example :

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\mathbf{\frac{1}{8}}$	$\mathbf{0}$	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	$\mathbf{0}$	$\mathbf{0}$	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

Joint distribution function $F_{X,Y}(x,y) \equiv P(X \leq x, Y \leq y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$F_X(\cdot)$
$x=0$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$x=1$	$\frac{1}{8}$	$\mathbf{\frac{2}{8}}$	$\frac{3}{8}$	$\mathbf{\frac{4}{8}}$	$\frac{4}{8}$
$x=2$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{7}{8}$
$x=3$	$\frac{1}{8}$	$\mathbf{\frac{5}{8}}$	$\frac{7}{8}$	$\mathbf{1}$	1
$F_Y(\cdot)$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	1	1

QUESTION : Why is

$$P(1 < X \leq 3\,, 1 < Y \leq 3) \;=\; F(3,3) \,-\, F(1,3) \,-\, F(3,1) \,+\, F(1,1)\ ?$$

EXERCISE :

Roll a *four-sided die* (tetrahedron) *two* times.

(The sides are marked 1 , 2 , 3 , 4 .)

Suppose each of the four sides is equally likely to end facing down.

Suppose the *outcome* of a *single roll* is the side that faces *down* (!).

Define the random variables X and Y as

$X =$ result of the *first roll* , $Y =$ *sum* of the two rolls.

- What is a good choice of the *sample space* $\mathcal{S}$?
- How many outcomes are there in $\mathcal{S}$?
- List the values of the *joint probability mass function* $p_{X,Y}(x, y)$.
- List the values of the *joint cumulative distribution function* $F_{X,Y}(x, y)$.

EXERCISE :

Three balls are selected at random from a bag containing

2 *red* , 3 *green* , 4 *blue* balls .

Define the *random variables*

$$R(s) \quad = \quad \text{the number of } \textit{red} \text{ balls drawn,}$$

and

$$G(s) \quad = \quad \text{the number of } \textit{green} \text{ balls drawn .}$$

List the values of

- the *joint probability mass function* $p_{R,G}(r,g)$.
- the *marginal probability mass functions* $p_R(r)$ and $p_G(g)$.
- the *joint distribution function* $F_{R,G}(r,g)$.
- the *marginal distribution functions* $F_R(r)$ and $F_G(g)$.

Independent random variables

Two discrete random variables $X(s)$ and $Y(s)$ are *independent* if

$$P(X = x \ , \ Y = y) \ = \ P(X = x) \cdot P(Y = y) \ , \quad \text{for } \textit{all } x \text{ and } y \ ,$$

or, equivalently, if their *probability mass functions* satisfy

$$p_{X,Y}(x,y) \quad = \quad p_X(x) \ \cdot \ p_Y(y) \ , \quad \text{for } \textit{all } x \text{ and } y \ ,$$

or, equivalently, if the *events*

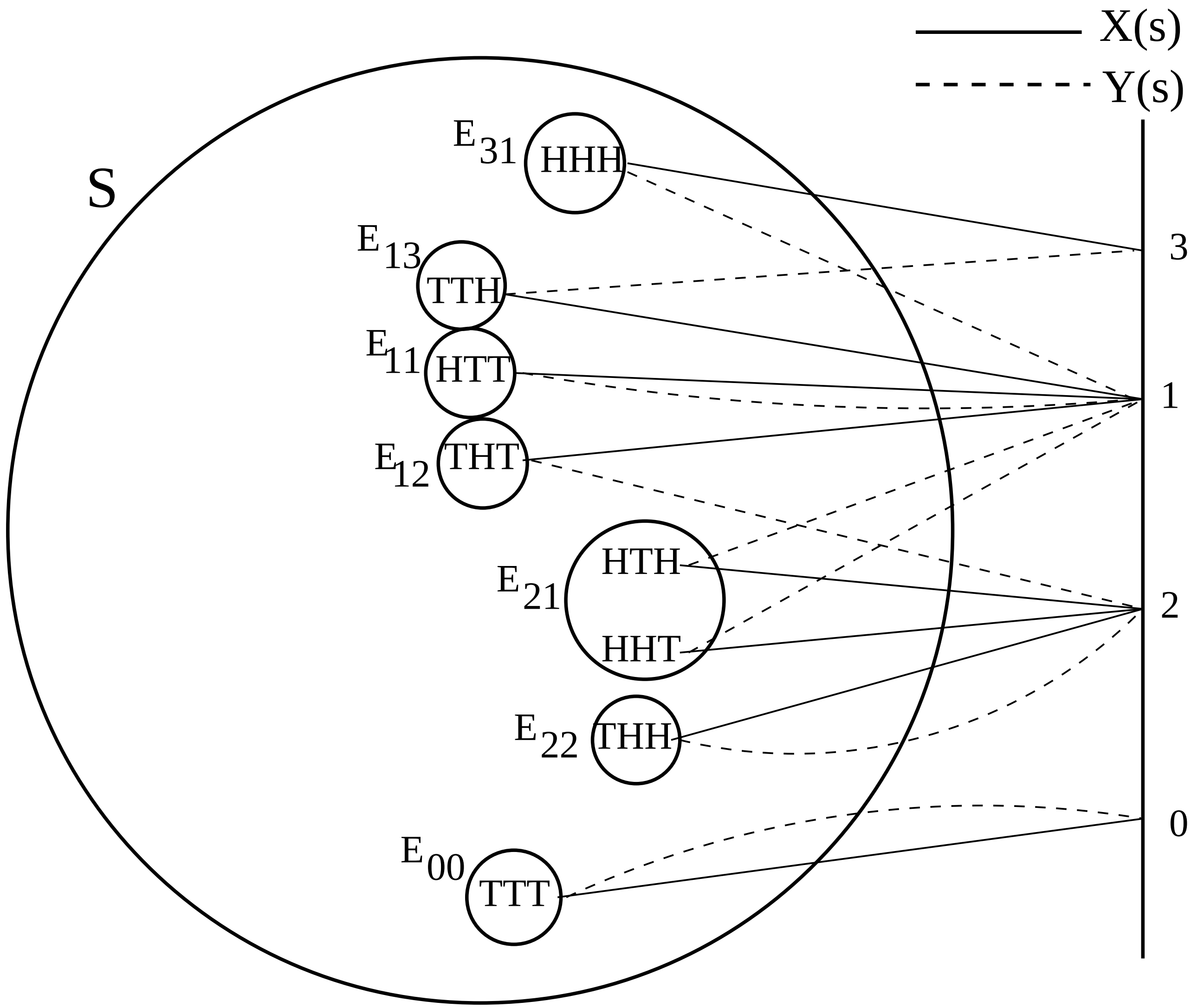
$$E_x \ \equiv \ X^{-1}(\{x\}) \quad \text{and} \quad E_y \ \equiv \ Y^{-1}(\{y\}) \ ,$$

are independent *in the sample space* $\mathcal{S}$, *i.e.*,

$$P(E_x E_y) \quad = \quad P(E_x) \cdot P(E_y) \ , \quad \text{for } \textit{all } x \text{ and } y \ .$$

NOTE :

- In the current *discrete* case, x and y are typically *integers* .
- $X^{-1}(\{x\}) \ \equiv \ \{ \ s \in \mathcal{S} \ : \ X(s) = x \ \}$.



Three tosses : $X(s) = \#$ Heads, $Y(s) =$ index 1^{st} H .

- What are the values of $p_X(2)$, $p_Y(1)$, $p_{X,Y}(2,1)$?
- Are X and Y *independent* ?

RECALL :

$X(s)$ and $Y(s)$ are *independent* if for all x and y :

$$p_{X,Y}(x,y) \quad = \quad p_X(x) \; \cdot \; p_Y(y) \; .$$

EXERCISE :

Roll a die two times in a row.

Let

$$X \text{ be the result of the } 1^{\text{st}} \text{ roll },$$

and

$$Y \text{ the result of the } 2^{\text{nd}} \text{ roll }.$$

Are X and Y *independent* , *i.e.*, is

$$p_{X,Y}(k,\ell) \; = \; p_X(k) \cdot p_Y(\ell), \qquad \text{for all } \; 1 \; \leq \; k,\ell \; \leq \; 6 \; \; ?$$

EXERCISE :

Are these random variables X and Y *independent* ?

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

EXERCISE : Are these random variables X and Y *independent* ?

Joint probability mass function $p_{X,Y}(x, y)$

	$y = 1$	$y = 2$	$y = 3$	$p_X(x)$
$x = 1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x = 2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x = 3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Joint distribution function $F_{X,Y}(x, y) \equiv P(X \le x, Y \le y)$

	$y = 1$	$y = 2$	$y = 3$	$F_X(x)$
$x = 1$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{2}$
$x = 2$	$\frac{5}{9}$	$\frac{25}{36}$	$\frac{5}{6}$	$\frac{5}{6}$
$x = 3$	$\frac{2}{3}$	$\frac{5}{6}$	1	1
$F_Y(y)$	$\frac{2}{3}$	$\frac{5}{6}$	1	1

QUESTION : Is $F_{X,Y}(x, y) \;=\; F_X(x) \cdot F_Y(y)$?

PROPERTY :

The *joint distribution function* of *independent* random variables X and Y satisfies

$$F_{X,Y}(x,y) \quad = \quad F_X(x) \cdot F_Y(y) \ , \qquad \text{for all } x, y \ .$$

PROOF :

$$\begin{aligned}
F_{X,Y}(x_k, y_\ell) \quad &= \quad P(X \leq x_k \ , \ Y \leq y_\ell) \\
&= \quad \sum_{i \leq k} \ \sum_{j \leq \ell} \ p_{X,Y}(x_i, y_j) \\
&= \quad \sum_{i \leq k} \ \sum_{j \leq \ell} \ p_X(x_i) \cdot p_Y(y_j) \qquad \text{(by independence)} \\
&= \quad \sum_{i \leq k} \ \{ \ p_X(x_i) \cdot \sum_{j \leq \ell} \ p_Y(y_j) \ \} \\
&= \quad \{ \sum_{i \leq k} \ p_X(x_i) \ \} \ \cdot \ \{ \sum_{j \leq \ell} \ p_Y(y_j) \ \} \\
&= \quad F_X(x_k) \cdot F_Y(y_\ell) \ .
\end{aligned}$$

Conditional distributions

Let X and Y be discrete random variables with *joint probability mass function*

$$p_{X,Y}(x,y) \ .$$

For given x and y, let

$$E_x = X^{-1}(\{x\}) \qquad \text{and} \qquad E_y = Y^{-1}(\{y\}) \ ,$$

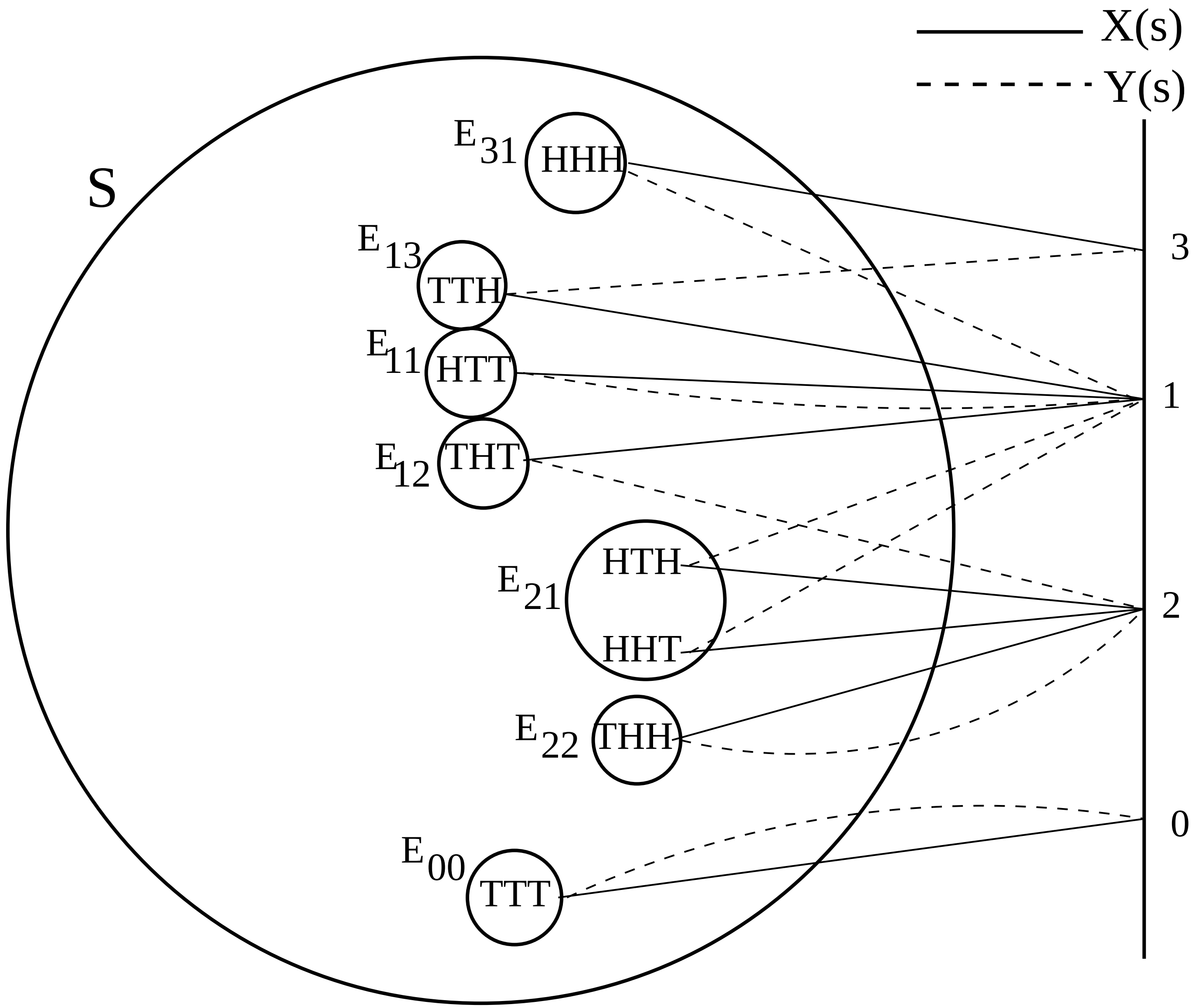
be their corresponding *events* in the sample space $\mathcal{S}$.

Then

$$P(E_x|E_y) \equiv \frac{P(E_xE_y)}{P(E_y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)} \ .$$

Thus it is natural to define the *conditional probability mass function*

$$p_{X|Y}(x|y) \equiv P(X=x \mid Y=y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \ .$$



Three tosses : $X(s) = \#\ \text{Heads},\ \ Y(s) = \text{index } 1^{\text{st}}\ \ H$.

- What are the values of $P(X = 2 \,|\, Y = 1)$ and $P(Y = 1 \,|\, X = 2)$?

EXAMPLE : (3 tosses : $X(s) = \#$ Heads, $Y(s) =$ index 1^{st} H.)

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

Conditional probability mass function $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$.

	$y=0$	$y=1$	$y=2$	$y=3$
$x=0$	1	0	0	0
$x=1$	0	$\frac{2}{8}$	$\frac{4}{8}$	1
$x=2$	0	$\frac{4}{8}$	$\frac{4}{8}$	0
$x=3$	0	$\frac{2}{8}$	0	0
	1	1	1	1

EXERCISE : Also construct the Table for $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$.

EXAMPLE :

Joint probability mass function $p_{X,Y}(x,y)$

	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x=2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x=3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Conditional probability mass function $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$.

	$y=1$	$y=2$	$y=3$
$x=1$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$x=2$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$x=3$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
	1	1	1

QUESTION : What does the last Table tell us?

EXERCISE : Also construct the Table for $P(Y=y|X=x)$.

Expectation

The *expected value* of a discrete random variable X is

$$E[X] \quad \equiv \quad \sum_k x_k \cdot P(X = x_k) \quad = \quad \sum_k x_k \cdot p_X(x_k) \ .$$

Thus $E[X]$ represents the *weighted average value* of X .

($E[X]$ is also called the *mean* of X .)

EXAMPLE : The *expected value* of *rolling a die* is

$$E[X] \quad = \quad 1 \cdot \tfrac{1}{6} \ + \ 2 \cdot \tfrac{1}{6} \ + \ \cdots \ + \ 6 \cdot \tfrac{1}{6} \quad = \quad \tfrac{1}{6} \cdot \textstyle\sum_{k=1}^{6} k \quad = \quad \tfrac{7}{2} \ .$$

EXERCISE : Prove the following :

- $E[aX] \quad = \quad a \ E[X]$,
- $E[aX + b] \quad = \quad a \ E[X] \ + \ b$.

EXAMPLE : Toss a coin until "Heads" occurs. Then

$$\mathcal{S} \ = \ \{H \ , \ TH \ , \ TTH \ , \ TTTH \ , \ \cdots \} \ .$$

The *random variable* X is the *number of tosses* until "Heads" occurs :

$$X(H) \ = \ 1 \quad , \quad X(TH) \ = \ 2 \quad , \quad X(TTH) \ = \ 3 \ .$$

Then

$$E[X] \ = \ 1 \cdot \frac{1}{2} \ + \ 2 \cdot \frac{1}{4} \ + \ 3 \cdot \frac{1}{8} \ + \ \cdots \ = \ \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{2^k} \ = \ 2 \ .$$

n	$\sum_{k=1}^{n} \ k/2^k$
1	0.50000000
2	1.00000000
3	1.37500000
10	1.98828125
40	1.99999999

REMARK :

Perhaps using $\mathcal{S}_n = \{$all sequences of n tosses$\}$ is better $\cdots$

The expected value of a *function of a random variable* is

$$E[g(X)] \equiv \sum_k g(x_k)\, p(x_k) .$$

EXAMPLE :

The *pay-off* of rolling a die is $\$k^2$, where k is the side facing up.

What should the *entry fee* be for the betting to break even?

SOLUTION : Here $g(X) = X^2$, and

$$E[g(X)] = \sum_{k=1}^{6} k^2 \frac{1}{6} = \frac{1}{6} \frac{6(6+1)(2 \cdot 6+1)}{6} = \frac{91}{6} \cong \$15.17 .$$

The expected value of a function of *two* random variables is

$$E[g(X,Y)] \equiv \sum_k \sum_\ell g(x_k, y_\ell)\, p(x_k, y_\ell) .$$

EXAMPLE :

	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x=2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x=3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

$$E[X] = 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{3} + 3 \cdot \tfrac{1}{6} = \mathbf{\tfrac{5}{3}} ,$$

$$E[Y] = 1 \cdot \tfrac{2}{3} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} = \mathbf{\tfrac{3}{2}} ,$$

$$\begin{aligned} E[XY] = & \ 1 \cdot \tfrac{1}{3} + 2 \cdot \tfrac{1}{12} + 3 \cdot \tfrac{1}{12} \\ & + 2 \cdot \tfrac{2}{9} + 4 \cdot \tfrac{1}{18} + 6 \cdot \tfrac{1}{18} \\ & + 3 \cdot \tfrac{1}{9} + 6 \cdot \tfrac{1}{36} + 9 \cdot \tfrac{1}{36} = \mathbf{\tfrac{5}{2}} . \end{aligned}$$

(So ?)

PROPERTY :

- If X and Y are *independent* then $E[XY] = E[X]\,E[Y]$.

PROOF :

$$\begin{aligned}
E[XY] &= \textstyle\sum_k \sum_\ell \ x_k \ y_\ell \ p_{X,Y}(x_k, y_\ell) \\
&= \textstyle\sum_k \sum_\ell \ x_k \ y_\ell \ p_X(x_k) \ p_Y(y_\ell) \qquad \text{(by independence)} \\
&= \textstyle\sum_k \{ \ x_k \ p_X(x_k) \sum_\ell \ y_\ell \ p_Y(y_\ell) \} \\
&= \{ \textstyle\sum_k \ x_k \ p_X(x_k) \} \cdot \{ \sum_\ell \ y_\ell \ p_Y(y_\ell) \} \\
&= E[X] \cdot E[Y] \ .
\end{aligned}$$

EXAMPLE : See the preceding example !

PROPERTY : $E[X+Y] \ = \ E[X] \ + \ E[Y]$. (**Always** !)

PROOF :

$$\begin{aligned} E[X+Y] &= \textstyle\sum_k \sum_\ell \ (x_k + y_\ell) \ p_{X,Y}(x_k, y_\ell) \\ &= \textstyle\sum_k \sum_\ell \ x_k \ p_{X,Y}(x_k, y_\ell) \ + \ \sum_k \sum_\ell \ y_\ell \ p_{X,Y}(x_k, y_\ell) \\ &= \textstyle\sum_k \sum_\ell \ x_k \ p_{X,Y}(x_k, y_\ell) \ + \ \sum_\ell \sum_k \ y_\ell \ p_{X,Y}(x_k, y_\ell) \\ &= \textstyle\sum_k \{x_k \sum_\ell \ p_{X,Y}(x_k, y_\ell)\} \ + \ \sum_\ell \{ \ y_\ell \sum_k \ p_{X,Y}(x_k, y_\ell)\} \\ &= \textstyle\sum_k \{x_k \ p_X(x_k)\} \ + \ \sum_\ell \{y_\ell \ p_Y(y_\ell)\} \\ &= E[X] \ + \ E[Y] \ . \end{aligned}$$

NOTE : X and Y need not be independent !

EXERCISE :

Probability mass function $p_{X,Y}(x,y)$

	$y=6$	$y=8$	$y=10$	$p_X(x)$
$x=1$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$x=2$	0	$\frac{1}{5}$	0	$\frac{1}{5}$
$x=3$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$p_Y(y)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	1

Show that

- $E[X] = 2 \quad , \quad E[Y] = 8 \quad , \quad E[XY] = 16$

- X and Y are *not* independent

Thus if

$$E[XY] = E[X]\,E[Y] \,,$$

then it does not necessarily follow that X and Y are independent !

Variance and Standard Deviation

Let X have *mean*

$$\mu \;=\; E[X]\ .$$

Then the *variance* of X is

$$Var(X) \;\equiv\; E[\,(X-\mu)^2] \;\equiv\; \sum_k \,(x_k-\mu)^2\, p(x_k)\ ,$$

which is the average weighted *square distance* from the mean.

We have

$$\begin{aligned} Var(X) &= E[X^2 \;-\; 2\mu X \;+\; \mu^2] \\ &= E[X^2] \;-\; 2\mu E[X] \;+\; \mu^2 \\ &= E[X^2] \;-\; 2\mu^2 \;+\; \mu^2 \\ &= E[X^2] \;-\; \mu^2\ . \end{aligned}$$

The *standard deviation* of X is

$$\sigma(X) \;\equiv\; \sqrt{Var(X)} \;=\; \sqrt{E[\,(X-\mu)^2]} \;=\; \sqrt{E[X^2]\;-\;\mu^2}\;.$$

which is the average weighted *distance* from the mean.

EXAMPLE : The *variance* of *rolling a die* is

$$Var(X) \;=\; \sum_{k=1}^{6}\,[k^2\cdot\frac{1}{6}\,]\;-\;\mu^2$$

$$=\;\frac{1}{6}\,\frac{6(6+1)(2\cdot 6+1)}{6}\;-\;(\frac{7}{2})^2\;=\;\frac{35}{12}\;.$$

The *standard deviation* is

$$\sigma \;=\; \sqrt{\frac{35}{12}} \;\cong\; 1.70\;.$$

Covariance

Let X and Y be random variables with *mean*

$$E[X] = \mu_X \quad , \quad E[Y] = \mu_Y \; .$$

Then the *covariance* of X and Y is defined as

$$Cov(X,Y) \equiv E[\,(X-\mu_X)\,(Y-\mu_Y)\,] = \sum_{k,\ell} (x_k-\mu_X)\,(y_\ell-\mu_Y)\,p(x_k,y_\ell)\; .$$

We have

$$\begin{aligned}
Cov(X,Y) &= E[\,(X-\mu_X)\,(Y-\mu_Y)\,] \\
&= E[XY - \mu_X Y - \mu_Y X \; + \; \mu_X\mu_Y] \\
&= E[XY] \; - \; \mu_X\mu_Y \; - \; \mu_Y\mu_X \; + \; \mu_X\mu_Y \\
&= E[XY] \; - \; E[X]\,E[Y]\; .
\end{aligned}$$

We defined

$$
\begin{aligned}
Cov(X,Y) &\equiv E[\,(X-\mu_X)\,(Y-\mu_Y)\,] \\
&= \sum_{k,\ell} (x_k-\mu_X)\,(y_\ell-\mu_Y)\,p(x_k,y_\ell) \\
&= E[XY] \;-\; E[X]\,E[Y]\,.
\end{aligned}
$$

NOTE :

$Cov(X,Y)$ measures "*concordance*" or "*coherence*" of X and Y :

- If $X > \mu_X$ when $Y > \mu_Y$ and $X < \mu_X$ when $Y < \mu_Y$ then

$$Cov(X,Y) \;>\; 0\,.$$

- If $X > \mu_X$ when $Y < \mu_Y$ and $X < \mu_X$ when $Y > \mu_Y$ then

$$Cov(X,Y) \;<\; 0\,.$$

EXERCISE : Prove the following :

- $Var(aX + b) \quad = \quad a^2 \; Var(X)$,

- $Cov(X, Y) \quad = \quad Cov(Y, X)$,

- $Cov(cX, Y) \quad = \quad c \, Cov(X, Y)$,

- $Cov(X, cY) \quad = \quad c \, Cov(X, Y)$,

- $Cov(X + Y, Z) \quad = \quad Cov(X, Z) \; + \; Cov(Y, Z)$,

- $Var(X + Y) \quad = \quad Var(X) \; + \; Var(Y) \; + \; 2 \, Cov(X, Y)$.

PROPERTY :

If X and Y are *independent* then $Cov(X,Y) = 0$.

PROOF :

We have already shown (with $\mu_X \equiv E[X]$ and $\mu_Y \equiv E[Y]$) that

$$Cov(X,Y) \equiv E[\,(X-\mu_X)\,(Y-\mu_Y)\,] = E[XY] - E[X]\,E[Y]\,,$$

and that if X and Y are *independent* then

$$E[XY] = E[X]\,E[Y]\,.$$

from which the result follows.

EXERCISE : (already used earlier $\cdots$)

Probability mass function $p_{X,Y}(x,y)$

	$y = 6$	$y = 8$	$y = 10$	$p_X(x)$
$x = 1$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$x = 2$	0	$\frac{1}{5}$	0	$\frac{1}{5}$
$x = 3$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$p_Y(y)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	1

Show that

- $E[X] \ = \ 2 \quad , \quad E[Y] \ = \ 8 \quad , \quad E[XY] \ = \ 16$
- $Cov(X,Y) \ = \ E[XY] \ - \ E[X] \ E[Y] \ = \ 0$
- X and Y are *not* independent

Thus if

$$Cov(X,Y) \ = \ 0 \ ,$$

then it does not necessarily follow that X and Y are independent !

PROPERTY :

If X and Y are *independent* then

$$Var(X+Y) = Var(X) + Var(Y) .$$

PROOF :

We have already shown (in an exercise !) that

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y) ,$$

and that if X and Y are *independent* then

$$Cov(X,Y) = 0 ,$$

from which the result follows.

EXERCISE :

Compute

$$E[X] \ , \quad E[Y] \ , \quad E[X^2] \ , \quad E[Y^2]$$

$$E[XY] \ , \quad Var(X) \ , \quad Var(Y)$$

$$Cov(X,Y)$$

for

Joint probability mass function $p_{X,Y}(x,y)$

	$y = 0$	$y = 1$	$y = 2$	$y = 3$	$p_X(x)$
$x = 0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x = 1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x = 2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x = 3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

EXERCISE :

Compute

$$E[X] \quad , \quad E[Y] \quad , \quad E[X^2] \quad , \quad E[Y^2]$$

$$E[XY] \quad , \quad Var(X) \quad , \quad Var(Y)$$

$$Cov(X,Y)$$

for

Joint probability mass function $p_{X,Y}(x,y)$

	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x=2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x=3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

SPECIAL DISCRETE RANDOM VARIABLES

The Bernoulli Random Variable

A *Bernoulli trial* has only *two outcomes* , with probability

$$P(X = 1) \quad = \ p \ ,$$

$$P(X = 0) \quad = \ 1 - p \ ,$$

e.g., tossing a coin, winning or losing a game, $\cdots$.

We have

$$E[X] \ = \ 1 \cdot p \ + \ 0 \cdot (1 - p) \ = \ p \ ,$$

$$E[X^2] \ = \ 1^2 \cdot p \ + \ 0^2 \cdot (1 - p) \ = \ p \ ,$$

$$Var(X) \ = \ E[X^2] - E[X]^2 \ = \ p - p^2 \ = \ p(1 - p) \ .$$

NOTE : If p *is small* then $Var(X) \cong p$.

EXAMPLES :

- When $p = \frac{1}{2}$ ($e.g.$, for tossing a coin), we have

$$E[X] = p = \tfrac{1}{2} \quad , \quad Var(X) = p(1-p) = \tfrac{1}{4} .$$

- When *rolling a die* , with outcome k , $(1 \leq k \leq 6)$, let

$$X(k) = 1 \ \text{ if the roll resulted in a } \ six ,$$

and

$$X(k) = 0 \ \text{ if the roll did } \ not \ \text{ result in a } \ six .$$

Then

$$E[X] = p = \tfrac{1}{6} \quad , \quad Var(X) = p(1-p) = \tfrac{5}{36} .$$

- When $p = 0.01$, then

$$E[X] = 0.01 \quad , \quad Var(X) = 0.0099 \cong 0.01 .$$

The Binomial Random Variable

Perform a Bernoulli trial n times *in sequence* .

Assume the individual trials are *independent* .

An *outcome* could be

$$100011001010 \qquad\qquad (n = 12) \ ,$$

with probability

$$P(100011001010) \quad = \quad p^5 \cdot (1-p)^7 \ . \qquad\qquad (\text{ Why ? })$$

Let the X be the number of " *successes* " (*i.e.* 1's) .

For example,

$$X(100011001010) \quad = \quad 5 \ .$$

We have

$$P(X = 5) \quad = \quad \binom{12}{5} \cdot p^5 \cdot (1-p)^7 \ . \qquad\qquad (\text{ Why ? })$$

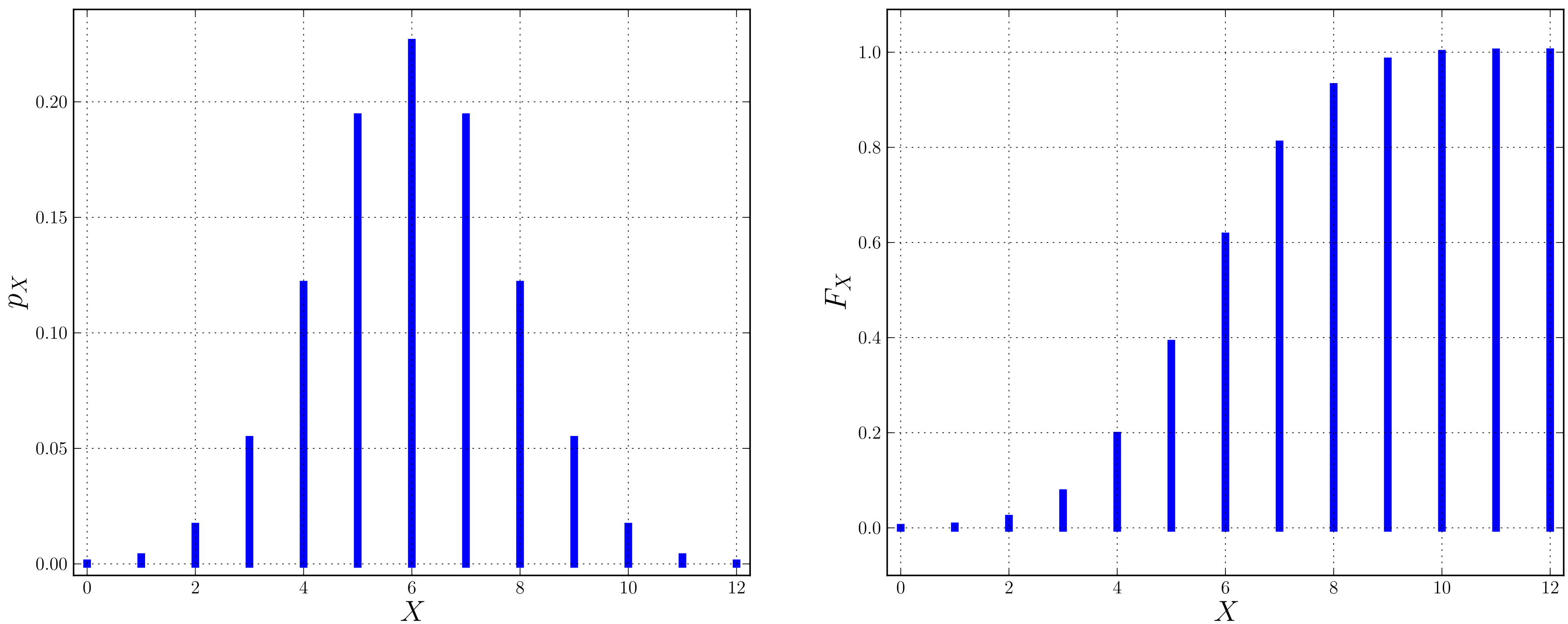
In general, for k successes in a sequence of n trials, we have

$$P(X = k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \ , \qquad\qquad (0 \leq k \leq n) \ .$$

EXAMPLE : Tossing a coin 12 times:

$$n = 12 \quad , \quad \mathbf{p = \tfrac{1}{2}}$$

k	$p_X(k)$	$F_X(k)$
0	1 / 4096	1 / 4096
1	12 / 4096	13 / 4096
2	66 / 4096	79 / 4096
3	220 / 4096	299 / 4096
4	495 / 4096	794 / 4096
5	792 / 4096	1586 / 4096
6	924 / 4096	2510 / 4096
7	792 / 4096	3302 / 4096
8	495 / 4096	3797 / 4096
9	220 / 4096	4017 / 4096
10	66 / 4096	4083 / 4096
11	12 / 4096	4095 / 4096
12	1 / 4096	4096 / 4096



The Binomial *mass* and *distribution* functions for $n = 12$, $p = \frac{1}{2}$

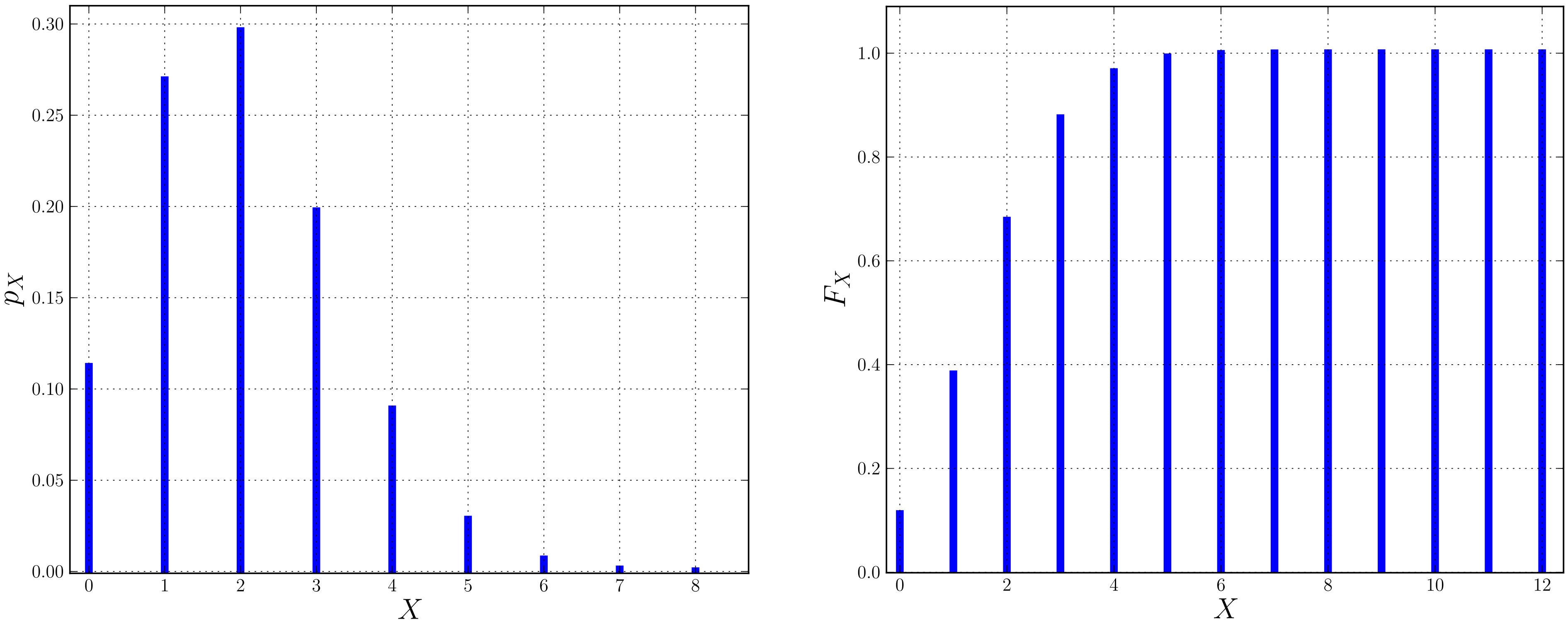
For k successes in a sequence of n trials :

$$P(X=k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \;, \qquad\qquad (0 \leq k \leq n) \;.$$

EXAMPLE : Rolling a die 12 times:

$$n = 12 \quad , \quad \mathbf{p = \frac{1}{6}}$$

k	$p_X(k)$	$F_X(k)$
0	0.1121566221	0.112156
1	0.2691758871	0.381332
2	0.2960935235	0.677426
3	0.1973956972	0.874821
4	0.0888280571	0.963649
5	0.0284249838	0.992074
6	0.0066324966	0.998707
7	0.0011369995	0.999844
8	0.0001421249	0.999986
9	0.0000126333	0.999998
10	0.0000007580	0.999999
11	0.0000000276	0.999999
12	0.0000000005	1.000000



The Binomial *mass* and *distribution* functions for $n = 12$, $p = \frac{1}{6}$

EXAMPLE :

In 12 *rolls of a die* write the outcome as, for example,

$$100011001010$$

where

$$1 \text{ denotes the roll resulted in a } six \ ,$$

and

$$0 \text{ denotes the roll did } not \text{ result in a } six \ .$$

As before, let X be the number of 1's in the outcome.

Then X represents the *number of sixes* in the 12 rolls.

Then, for example, using the preceding *Table* :

$$P(X = 5) \ \cong \ 2.8 \ \% \qquad , \qquad P(X \leq 5) \ \cong \ 99.2 \ \% \ .$$

EXERCISE : Show that from

$$P(X = k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \ ,$$

and

$$P(X = k+1) \quad = \quad \binom{n}{k+1} \cdot p^{k+1} \cdot (1-p)^{n-k-1} \ ,$$

it follows that

$$P(X = k+1) \quad = \quad c_k \cdot P(X = k) \ ,$$

where

$$c_k \quad = \quad \frac{n-k}{k+1} \cdot \frac{p}{1-p} \quad .$$

NOTE : This *recurrence formula* is an efficient and stable *algorithm* to compute the binomial probabilities :

$$P(X = 0) \ = \ (1-p)^n \ ,$$

$$P(X = k+1) \quad = \quad c_k \cdot P(X = k) \ , \qquad k = 0, 1, \cdots, n-1 \ .$$

Mean and variance of the Binomial random variable :

By definition, the *mean* of a Binomial random variable X is

$$E[X] \;=\; \sum_{k=0}^{n} k \cdot P(X=k) \;=\; \sum_{k=0}^{n} k \cdot \binom{n}{k} \; p^k \; (1-p)^{n-k} \;,$$

which can be shown to equal np .

An *easy way* to see this is as follows :

If in a *sequence* of n independent Bernoulli trials we let

$X_k \;=\;$ the outcome of the k^{th} Bernoulli trial , $\quad (X_k = 0 \text{ or } 1\ )$,

then

$$X \;\equiv\; X_1 \;+\; X_2 \;+\; \cdots \;+\; X_n \;,$$

is the *Binomial random variable* that *counts the successes* ” .

$$X \equiv X_1 + X_2 + \cdots + X_n$$

We know that

$$E[X_k] = p \ ,$$

so

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = np \ .$$

We already know that

$$Var(X_k) = E[X_k^2] - (E[X_k])^2 = p - p^2 = p(1-p) \ ,$$

so, since the X_k are *independent* , we have

$$Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_n) = np(1-p) \ .$$

NOTE : If p *is small* then $Var(X) \cong np$.

EXAMPLES :

- For 12 tosses of a *coin* , with *Heads* is *success*, we have

so
$$n = 12 \quad , \quad p = \frac{1}{2}$$
$$E[X] \ = \ np \ = \ 6 \quad , \quad Var(X) \ = \ np(1-p) \ = \ 3 \ .$$

- For 12 rolls of a *die* , with *six* is *success* , we have

so
$$n = 12 \quad , \quad p = \frac{1}{6}$$
$$E[X] \ = \ np \ = \ 2 \quad , \quad Var(X) \ = \ np(1-p) \ = \ 5/3 \ .$$

- If $n = 500$ and $p = 0.01$, then

$$E[X] \ = \ np \ = \ 5 \quad , \quad Var(X) \ = \ np(1-p) \ = \ 4.95 \ \cong \ 5 \ .$$

The Poisson Random Variable

The Poisson variable *approximates* the Binomial random variable :

$$P(X = k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \cong \quad e^{-\lambda} \;\cdot\; \frac{\lambda^k}{k!} \;,$$

when we take

$$\lambda \quad = \quad n\; p \quad (\textit{ the average number of successes }) \;.$$

This approximation is *accurate* if n is *large* and p *small* .

Recall that for the Binomial random variable

$E[X] = n\; p$, and $Var(X) = np(1-p) \;\cong\; np$ when p is small.

Indeed, for the Poisson random variable we will show that

$$E[X] = \lambda \quad \text{and} \quad Var(X) = \lambda \;.$$

A *stable* and *efficient* way to compute the Poisson probability

$$P(X = k) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^k}{k!} \ , \qquad k = 0, 1, 2, \ \cdots \ ,$$

$$P(X = k+1) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^{k+1}}{(k+1)!} \ ,$$

is to use the *recurrence relation*

$$P(X = 0) \ = \ e^{-\lambda} \ ,$$

$$P(X = k+1) \quad = \quad \frac{\lambda}{k+1} \cdot P(X = k) \ , \qquad k = 0, 1, 2, \ \cdots \ .$$

NOTE : Unlike the Binomial random variable, the Poisson random variable can have an *arbitrarily large* integer value k.

The Poisson random variable

$$P(X = k) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^k}{k!} \, , \qquad k = 0, 1, 2, \ \cdots \ ,$$

has (as shown later) : $\quad E[X] \ = \ \lambda \quad$ and $\quad Var(X) \ = \ \lambda$.

The Poisson *distribution function* is

$$F(k) \quad = \quad P(X \leq k) \quad = \quad \sum_{\ell=0}^{k} e^{-\lambda} \frac{\lambda^\ell}{\ell!} \quad = \quad e^{-\lambda} \sum_{\ell=0}^{k} \frac{\lambda^\ell}{\ell!} \, ,$$

with, as should be the case,

$$\lim_{k \to \infty} F(k) \quad = \quad e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \quad = \quad e^{-\lambda} \, e^{\lambda} \quad = \quad 1 \, .$$

(using the *Taylor series* from Calculus for e^λ) .

The Poisson random variable

$$P(X = k) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^k}{k!} \ , \qquad k = 0, 1, 2, \ \cdots \ ,$$

models the probability of k "*successes*" in a given "time" interval,

when the *average* number of successes is λ .

EXAMPLE : Suppose customers arrive at the rate of *six* per hour.
The probability that k customers arrive in a one-hour period is

$$P(k = 0) \quad = \quad e^{-6} \cdot \frac{6^0}{0!} \quad \cong \quad 0.0024 \ ,$$

$$P(k = 1) \quad = \quad e^{-6} \cdot \frac{6^1}{1!} \quad \cong \quad 0.0148 \ ,$$

$$P(k = 2) \quad = \quad e^{-6} \cdot \frac{6^2}{2!} \quad \cong \quad 0.0446 \ .$$

The probability that more than 2 customers arrive is

$$1 \ - \ (0.0024 \ + \ 0.0148 \ + \ 0.0446) \quad \cong \quad 0.938 \ .$$

$$p_{\text{Binomial}}(k) = \binom{n}{k} \; p^k \; (1-p)^{n-k} \quad \cong \quad p_{\text{Poisson}}(k) = e^{-\lambda} \; \frac{\lambda^k}{k!}$$

EXAMPLE : $\lambda = 6$ customers/hour.

For the Binomial take $n = 12$, $p = 0.5$ (0.5 customers/5 minutes) ,

so that indeed $np = \lambda$.

k	p_{Binomial}	p_{Poisson}	F_{Binomial}	F_{Poisson}	
0	0.0002	0.0024	0.0002	0.0024	
1	0.0029	0.0148	0.0031	0.0173	
2	0.0161	0.0446	0.0192	0.0619	
3	0.0537	0.0892	0.0729	0.1512	
4	0.1208	0.1338	0.1938	0.2850	
5	0.1933	0.1606	0.3872	0.4456	
6	0.2255	0.1606	0.6127	0.6063	
7	0.1933	0.1376	0.8061	0.7439	
8	0.1208	0.1032	0.9270	0.8472	
9	0.0537	0.0688	0.9807	0.9160	
10	0.0161	0.0413	0.9968	0.9573	
11	0.0029	0.0225	0.9997	0.9799	
12	0.0002	0.0112	1.0000	0.9911$\star$	Why not 1.0000 ?

Here the approximation is *not so good* $\cdots$

$$p_{\text{Binomial}}(k) = \binom{n}{k} \; p^k \; (1-p)^{n-k} \quad \cong \quad p_{\text{Poisson}}(k) = e^{-\lambda} \; \frac{\lambda^k}{k!}$$

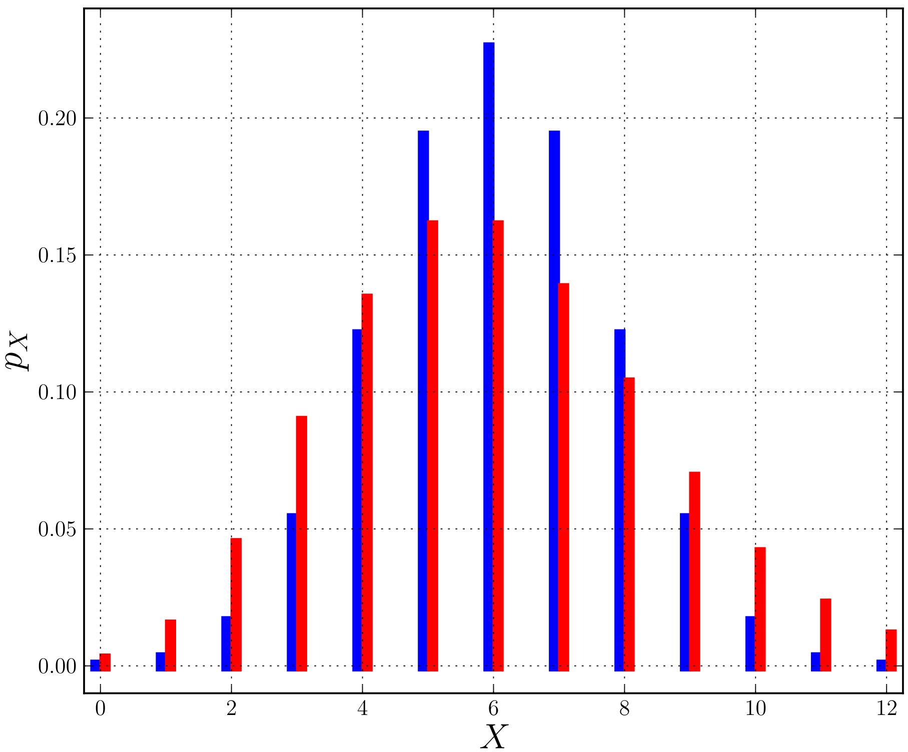
EXAMPLE : $\lambda = 6$ customers/hour.

For the Binomial take $n = 60$, $p = 0.1$ (0.1 customers/minute) ,

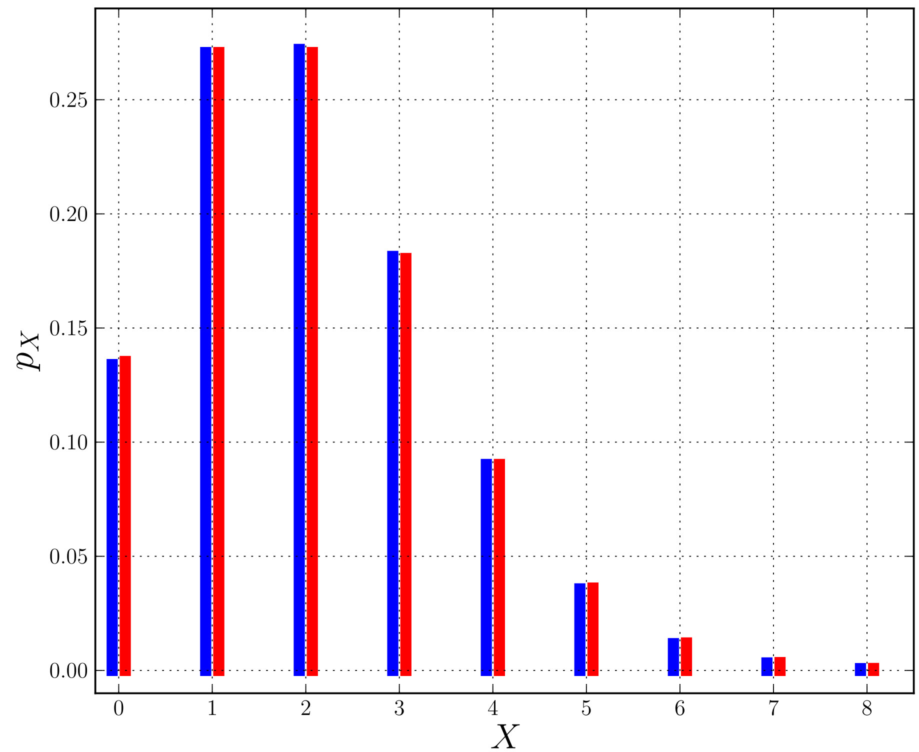
so that indeed $np = \lambda$.

k	p_{Binomial}	p_{Poisson}	F_{Binomial}	F_{Poisson}
0	0.0017	0.0024	0.0017	0.0024
1	0.0119	0.0148	0.0137	0.0173
2	0.0392	0.0446	0.0530	0.0619
3	0.0843	0.0892	0.1373	0.1512
4	0.1335	0.1338	0.2709	0.2850
5	0.1662	0.1606	0.4371	0.4456
6	0.1692	0.1606	0.6064	0.6063
7	0.1451	0.1376	0.7515	0.7439
8	0.1068	0.1032	0.8583	0.8472
9	0.0685	0.0688	0.9269	0.9160
10	0.0388	0.0413	0.9657	0.9573
11	0.0196	0.0225	0.9854	0.9799
12	0.0089	0.0112	0.9943	0.9911
13	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Here the approximation is *better* $\cdots$



$n = 12$, $p = \frac{1}{2}$, $\lambda = 6$



$n = 200$, $p = 0.01$, $\lambda = 2$

The Binomial (*blue*) and Poisson (*red*) probability mass functions.

For the case $n = 200$, $p = 0.01$, the approximation is very good !

For the *Binomial* random variable we found

$$E[X] \quad = \quad np \qquad \text{and} \qquad Var(X) \quad = \quad np(1-p) \ ,$$

while for the *Poisson* random variable, with $\lambda = np$ we will show

$$E[X] \quad = \quad np \qquad \text{and} \qquad Var(X) \quad = \quad np \ .$$

Note again that

$$np(1-p) \quad \cong \quad np \quad , \qquad \text{when } p \text{ is } \textit{small} \ .$$

EXAMPLE : In the preceding two *Tables* we have

n=12 , p=0.5

	Binomial	Poisson
$E[X]$	6.0000	6.0000
$Var[X]$	3.0000	6.0000
$\sigma[X]$	1.7321	2.4495

n=60 , p=0.1

	Binomial	Poisson
$E[X]$	6.0000	6.0000
$Var[X]$	5.4000	6.0000
$\sigma[X]$	2.3238	2.4495

FACT : (*The Method of Moments*)

By *Taylor expansion* of e^{tX} about $t = 0$, we have

$$\psi(t) \quad \equiv \quad E[e^{tX}] \quad = \quad E\Big[\, 1 \; + \; tX \; + \; \frac{t^2 \; X^2}{2!} \; + \; \frac{t^3 \; X^3}{3!} \; + \; \cdots \,\Big]$$

$$= \quad 1 \quad + \quad t \; E[X] \quad + \quad \frac{t^2}{2!} \, E[X^2] \quad + \quad \frac{t^3}{3!} \, E[X^3] \quad + \quad \cdots \quad .$$

It follows that

$$\psi'(0) \;=\; E[X] \qquad , \qquad \psi''(0) \;=\; E[X^2] \; . \qquad (\text{ Why ? })$$

This sometimes *facilitates computing the mean*

$$\mu \quad = \quad E[X] \; ,$$

and the variance

$$Var(X) \quad = \quad E[X^2] \quad - \quad \mu^2 \; .$$

APPLICATION : The *Poisson mean* and *variance* :

$$\psi(t) \equiv E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} P(X=k) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)} .$$

Here $\psi'(t) = \lambda e^t e^{\lambda(e^t-1)}$

$$\psi''(t) = \lambda \left[\lambda (e^t)^2 + e^t\right] e^{\lambda(e^t-1)} \qquad (\text{ Check ! })$$

so that $E[X] = \psi'(0) = \lambda$

$$E[X^2] = \psi''(0) = \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$Var(X) = E[X^2] - E[X]^2 = \lambda .$$

EXAMPLE : *Defects* in a wire occur at the rate of *one per* 10 *meter*, with a *Poisson distribution* :

$$P(X = k) \quad = \quad e^{-\lambda} \ \cdot \ \frac{\lambda^k}{k!} \ , \qquad k = 0, 1, 2, \ \cdots \ .$$

What is the probability that :

- A 12-meter roll has at *no* defects?

  ANSWER : Here $\lambda = 1.2$, and $P(X = 0) \quad = \quad e^{-\lambda} \quad = \quad 0.3012$.

- A 12-meter roll of wire has *one* defect?

  ANSWER : With $\lambda = 1.2$, $P(X = 1) \quad = \quad e^{-\lambda} \cdot \lambda \quad = \quad 0.3614$.

- Of *five* 12-meter rolls *two* have *one* defect and *three* have *none*?

  ANSWER : $\binom{5}{3} \cdot 0.3012^3 \cdot 0.3614^2 \quad = \quad 0.0357$. (Why ?)

EXERCISE :

Defects in a certain wire occur at the rate of one per 10 meter.
Assume the defects have a Poisson distribution.
What is the probability that :

- a 20-meter wire has no defects?

- a 20-meter wire has at most 2 defects?

EXERCISE :

Customers arrive at a counter at the rate of 8 per hour.
Assume the arrivals have a Poisson distribution.
What is the probability that :

- no customer arrives in 15 minutes?

- two customers arrive in a period of 30 minutes?

CONTINUOUS RANDOM VARIABLES

DEFINITION : A *continuous random variable* is a *function* $X(s)$ from an *uncountably infinite* sample space $\mathcal{S}$ to the real numbers $\mathbb{R}$,

$$X(\cdot) \quad : \quad \mathcal{S} \quad \rightarrow \quad \mathbb{R} \ .$$

EXAMPLE :

Rotate a *pointer* about a pivot in a plane (like a hand of a clock).

The *outcome* is the *angle* where it stops : $2\pi\theta$, where $\theta \in (0,1]$.

A good *sample space* is all values of θ , *i.e.* $\mathcal{S} \ = \ (0,1]$.

A very simple example of a *continuous random variable* is $X(\theta) = \theta$.

Suppose *any outcome*, *i.e.*, any value of θ is "equally likely".

What are the values of

$$P(0 \ < \ \theta \ \le \ \tfrac{1}{2}) \quad , \quad P(\tfrac{1}{3} \ < \ \theta \ \le \ \tfrac{1}{2}) \quad , \quad P(\theta = \tfrac{1}{\sqrt{2}}) \ ?$$

The (*cumulative*) *probability distribution function* is defined as

$$F_X(x) \;\equiv\; P(X \;\leq x)\;.$$

Thus

$$F_X(b) \;-\; F_X(a) \;\equiv\; P(a < X \;\leq b)\;.$$

We must have

$$F_X(-\infty) \;=\; 0 \qquad \text{and} \qquad F_X(\infty) \;=\; 1\;,$$

i.e.,

$$\lim_{x\to-\infty} F_X(x) \;=\; 0\;,$$

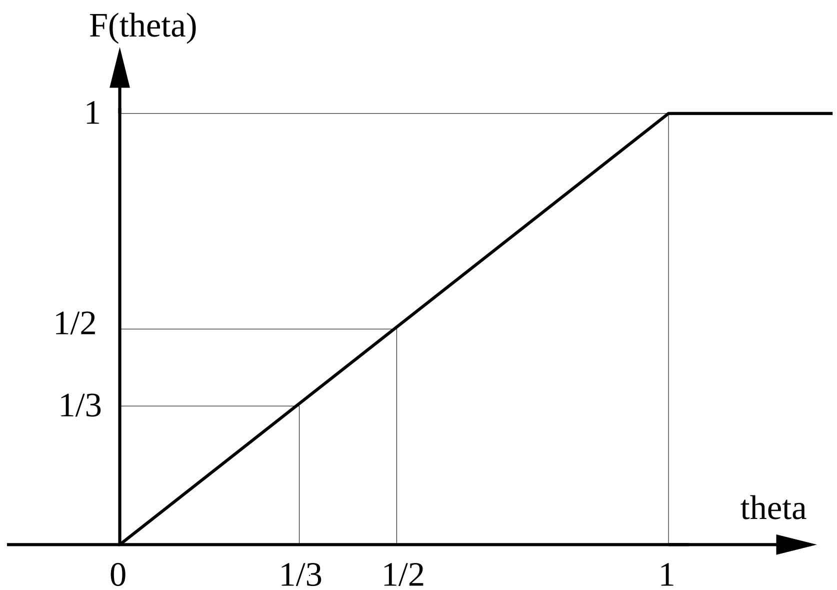
and

$$\lim_{x\to\infty} F_X(x) \;=\; 1\;.$$

Also, $F_X(x)$ is a *non-decreasing* function of x. (Why ?)

NOTE : All the above is *the same* as for *discrete* random variables !

EXAMPLE : In the "*pointer example*", where $X(\theta) = \theta$, we have the *probability distribution function*



Note that

$$F(\tfrac{1}{3}) \equiv P(X \leq \tfrac{1}{3}) = \tfrac{1}{3} \quad , \quad F(\tfrac{1}{2}) \equiv P(X \leq \tfrac{1}{2}) = \tfrac{1}{2} \ ,$$

$$P(\tfrac{1}{3} < X \leq \tfrac{1}{2}) = F(\tfrac{1}{2}) - F(\tfrac{1}{3}) = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6} \ .$$

QUESTION : What is $P(\frac{1}{3} \leq X \leq \frac{1}{2})$?

The *probability density function* is the *derivative* of the probability distribution function :

$$f_X(x) \quad \equiv \quad F_X'(x) \quad \equiv \quad \frac{d}{dx}\, F_X(x) \ .$$

EXAMPLE : In the "*pointer example* "

$$F_X(x) \quad = \quad \begin{cases} 0 \ , & x \leq 0 \\ x \ , & 0 < x \leq 1 \\ 1 \ , & 1 < x \end{cases}$$

Thus

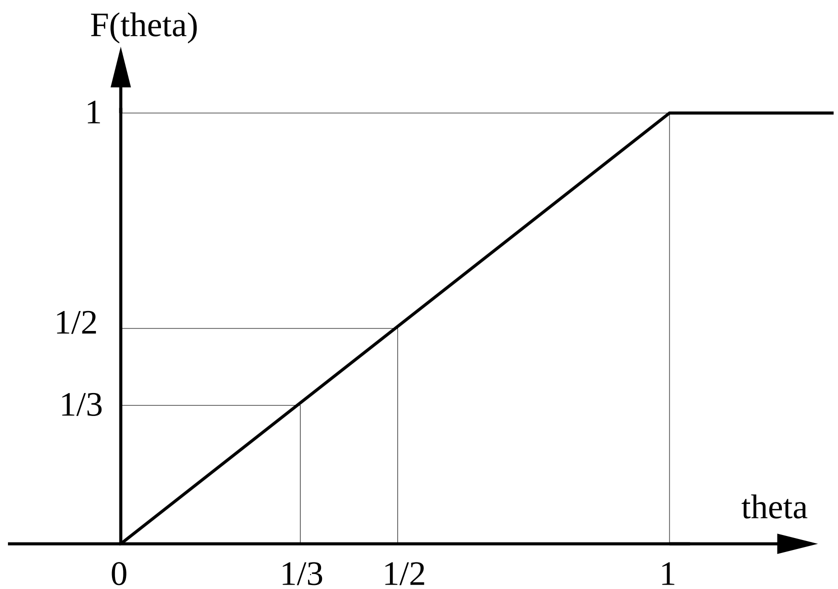
$$f_X(x) \quad = \quad F_X'(x) \quad = \quad \begin{cases} 0 \ , & x \leq 0 \\ 1 \ , & 0 < x \leq 1 \\ 0 \ , & 1 < x \end{cases}$$

NOTATION : When it is clear what X is then we also write

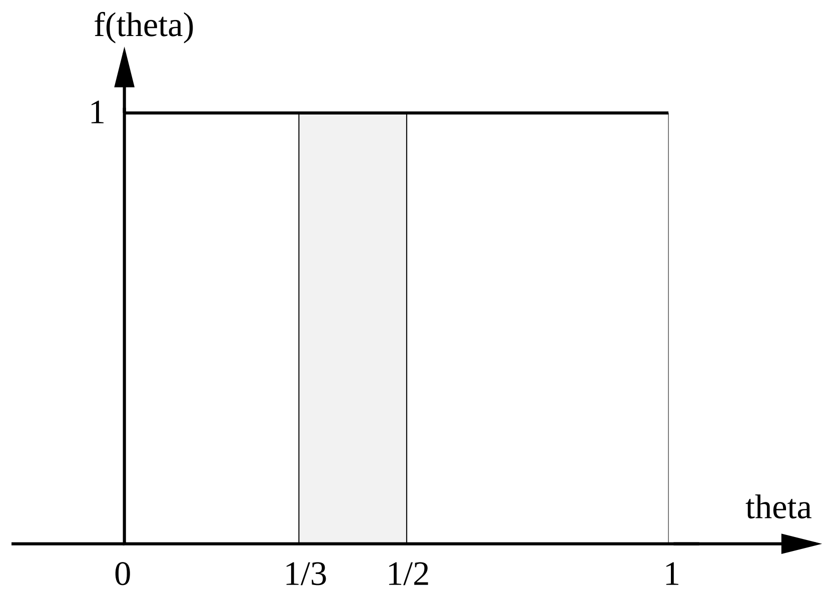
$$f(x) \quad \text{for} \quad f_X(x) \ , \qquad \text{and} \qquad F(x) \quad \text{for} \quad F_X(x) \ .$$

EXAMPLE : (continued $\cdots$)

$$F(x) = \begin{cases} 0\ , & x \le 0 \\ x\ , & 0 < x \le 1 \\ 1\ , & 1 < x \end{cases} \quad , \quad f(x) = \begin{cases} 0\ , & x \le 0 \\ 1\ , & 0 < x \le 1 \\ 0\ , & 1 < x \end{cases}$$



Distribution function



Density function

NOTE :

$$P(\frac{1}{3} < X \le \frac{1}{2}) \quad = \quad \int_{\frac{1}{3}}^{\frac{1}{2}} f(x)\ dx \quad = \quad \frac{1}{6} \quad = \quad \text{the shaded area .}$$

In general, from

$$f(x) \quad \equiv \quad F'(x) \ ,$$

with

$$F(-\infty) \ = \ 0 \qquad \text{and} \qquad F(\infty) \ = \ 1 \ ,$$

we have from Calculus the following *basic identities* :

$$\int_{-\infty}^{\infty} f(x)\ dx \ = \ \int_{-\infty}^{\infty} F'(x)\ dx \ = \ F(\infty) - F(-\infty) \ = \ 1 \ ,$$

$$\int_{-\infty}^{x} f(x)\ dx \ = \ F(x) - F(-\infty) \ = \ F(x) \ = \ P(X \le x) \ ,$$

$$\int_{a}^{b} f(x)\ dx \ = \ F(b) - F(a) \ = \ P(a < X \le b) \ ,$$

$$\int_{a}^{a} f(x)\ dx \ = \ F(a) - F(a) \ = \ 0 \ = \ P(X = a) \ .$$

EXERCISE : Draw *graphs* of the distribution and density functions

$$F(x) = \begin{cases} 0 \ , & x \leq 0 \\ 1 - e^{-x} \ , & x > 0 \end{cases} \quad , \quad f(x) = \begin{cases} 0 \ , & x \leq 0 \\ e^{-x} \ , & x > 0 \end{cases} \quad ,$$

and verify that

- $F(-\infty) \ = \ 0 \ , \qquad F(\infty) \ = \ 1 \ ,$
- $f(x) \ = \ F'(x) \ ,$
- $F(x) \ = \ \int_0^x \ f(x) \ dx \ ,$ (Why is *zero* as lower limit OK ?)
- $\int_0^\infty f(x) \ dx \ = \ 1 \ ,$
- $P(0 < X \leq 1) \ = \ F(1) - F(0) \ = \ F(1) \ = \ 1 - e^{-1} \ \cong \ 0.63 \ ,$
- $P(X > 1) \ = \ 1 - F(1) \ = \ e^{-1} \ \cong \ 0.37 \ ,$
- $P(1 < X \leq 2) \ = \ F(2) - F(1) \ = \ e^{-1} - e^{-2} \ \cong \ 0.23 \ .$

EXERCISE : For positive integer n, consider the density functions

$$f_n(x) = \begin{cases} cx^n(1-x^n) \ , & 0 \le x \le 1 \\ \\ 0 \ , & \text{otherwise} \end{cases}$$

- Determine the value of c in terms of n .
- Draw the graph of $f_n(x)$ for $n = 1, 2, 4, 8, 16$.
- Determine the distribution function $F_n(x)$.
- Draw the graph of $F_n(x)$ for $n = 1, 2, 3, 4, 8, 16$.
- Determine $P(0 \le X \le \frac{1}{2})$ in terms of n .
- What happens to $P(0 \le X \le \frac{1}{2})$ when n becomes large?
- Determine $P(\frac{9}{10} \le X \le 1)$ in terms of n .
- What happens to $P(\frac{9}{10} \le X \le 1)$ when n becomes large?

Joint distributions

A *joint probability density function* $f_{X,Y}(x,y)$ must satisfy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy \;=\; 1 \qquad (\text{“Volume”} = 1\ ).$$

The corresponding *joint probability distribution function* is

$$F_{X,Y}(x,y) \;=\; P(X \le x\ ,\ Y \le y) \;=\; \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy\ .$$

By Calculus we have $\displaystyle \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \;=\; f_{X,Y}(x,y)\ .$

Also,

$$P(a < X \le b\ ,\ c < Y \le d) \;=\; \int_{c}^{d}\int_{a}^{b} f_{X,Y}(x,y)\, dx\, dy\ .$$

EXAMPLE :

If

$$f_{X,Y}(x,y) \;=\; \begin{cases} 1 & \text{for } x \in (0,1] \text{ and } y \in (0,1] \ , \\ \\ 0 & \text{otherwise} \ , \end{cases}$$

then, for $x \in (0,1]$ and $y \in (0,1]$,

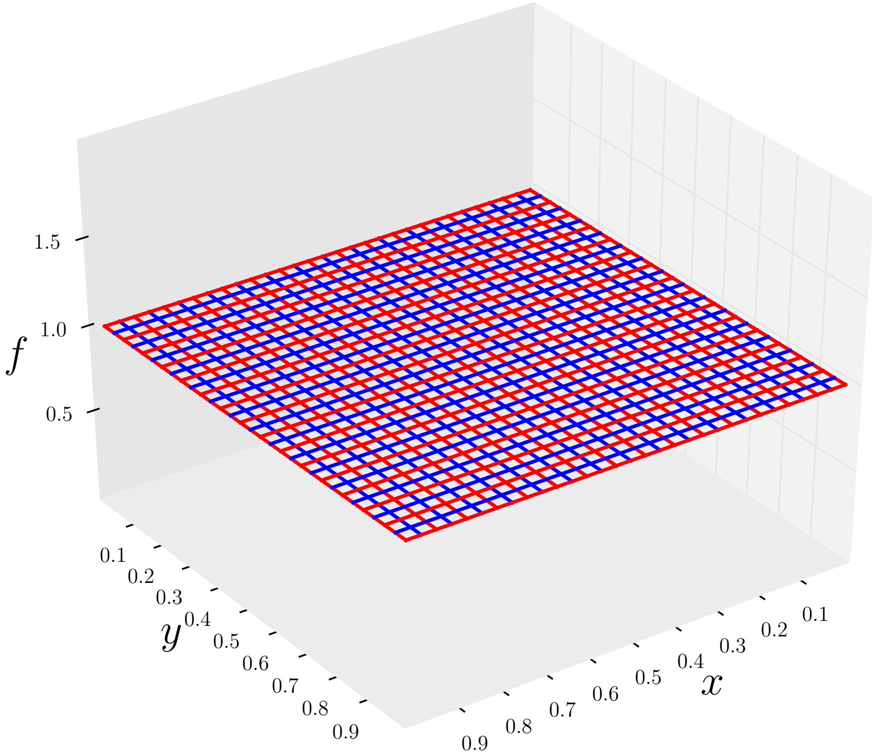
$$F_{X,Y}(x,y) \;=\; P(X \leq x \ , \ Y \leq y) \;=\; \int_0^y \int_0^x 1 \ dx \ dy \;=\; xy \ .$$

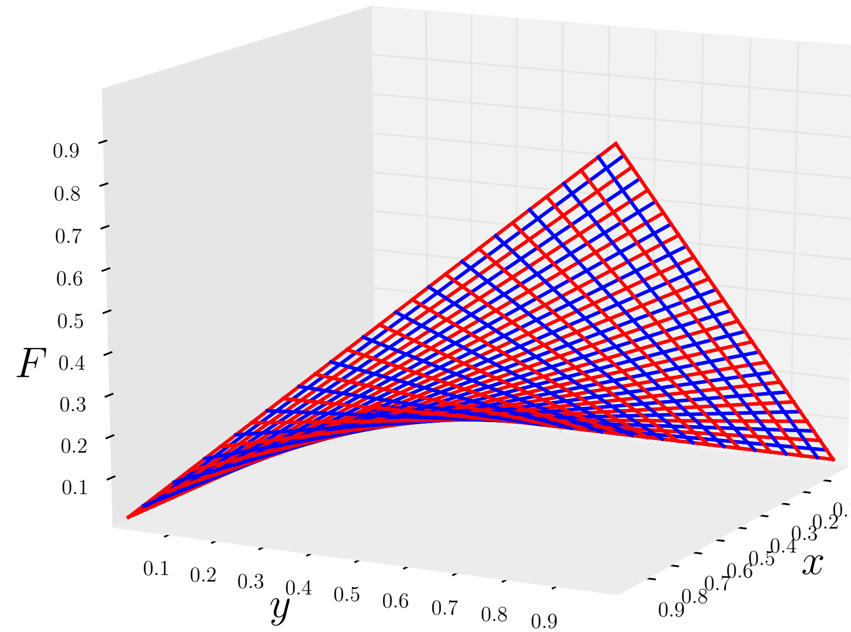
Thus

$$F_{X,Y}(x,y) \;=\; xy \ , \qquad \text{for } x \in (0,1] \text{ and } y \in (0,1] \ .$$

For example

$$P(\ X \leq \frac{1}{3} \ , \ Y \leq \frac{1}{2} \ ) \;=\; F_{X,Y}(\ \frac{1}{3} \ , \ \frac{1}{2} \ ) \;=\; \frac{1}{6} \ .$$





Also,

$$P\left(\frac{1}{3} \le X \le \frac{1}{2} \, , \, \frac{1}{4} \le Y \le \frac{3}{4} \right) \; = \; \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{3}}^{\frac{1}{2}} f(x,y) \, dx \, dy \; = \; \frac{1}{12} \, .$$

EXERCISE : Show that we can also compute this as follows :

$$F(\tfrac{1}{2},\tfrac{3}{4}) \; - \; F(\tfrac{1}{3},\tfrac{3}{4}) \; - \; F(\tfrac{1}{2},\tfrac{1}{4}) \; + \; F(\tfrac{1}{3},\tfrac{1}{4}) \; = \; \tfrac{1}{12} \, .$$

and explain why !

Marginal density functions

The *marginal density functions* are

$$f_X(x) \ = \ \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy \quad , \quad f_Y(y) \ = \ \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx \ .$$

with corresponding *marginal distribution functions*

$$F_X(x) \ \equiv \ P(X \le x) \ = \ \int_{-\infty}^{x} f_X(x)\, dx \ = \ \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx \ ,$$

$$F_Y(y) \ \equiv \ P(Y \le y) \ = \ \int_{-\infty}^{y} f_Y(y)\, dy \ = \ \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy \ .$$

By Calculus we have

$$\frac{dF_X(x)}{dx} \ = \ f_X(x) \quad , \quad \frac{dF_Y(y)}{dy} \ = \ f_Y(y) \ .$$

EXAMPLE : If

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{for } x \in (0,1] \text{ and } y \in (0,1] \ , \\ \\ 0 & \text{otherwise} \ , \end{cases}$$

then, for $x \in (0,1]$ and $y \in (0,1]$,

$$f_X(x) = \int_0^1 f_{X,Y}(x,y) \, dy = \int_0^1 1 \, dy = 1 \ ,$$

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y) \, dx = \int_0^1 1 \, dx = 1 \ ,$$

$$F_X(x) = P(X \leq x) = \int_0^x f_X(x) \, dx = x \ ,$$

$$F_Y(y) = P(Y \leq y) = \int_0^y f_Y(y) \, dy = y \ .$$

For example

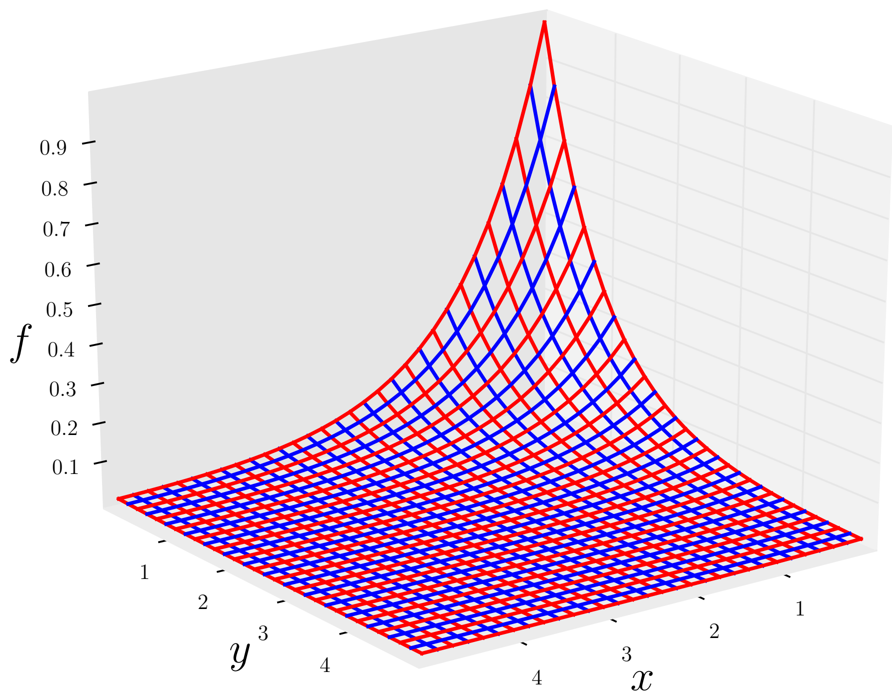
$$P(\ X \leq \tfrac{1}{3}\ ) = F_X(\tfrac{1}{3}) = \tfrac{1}{3} \quad , \quad P(\ Y \leq \tfrac{1}{2}\ ) = F_Y(\tfrac{1}{2}) = \tfrac{1}{2} \ .$$

EXERCISE :

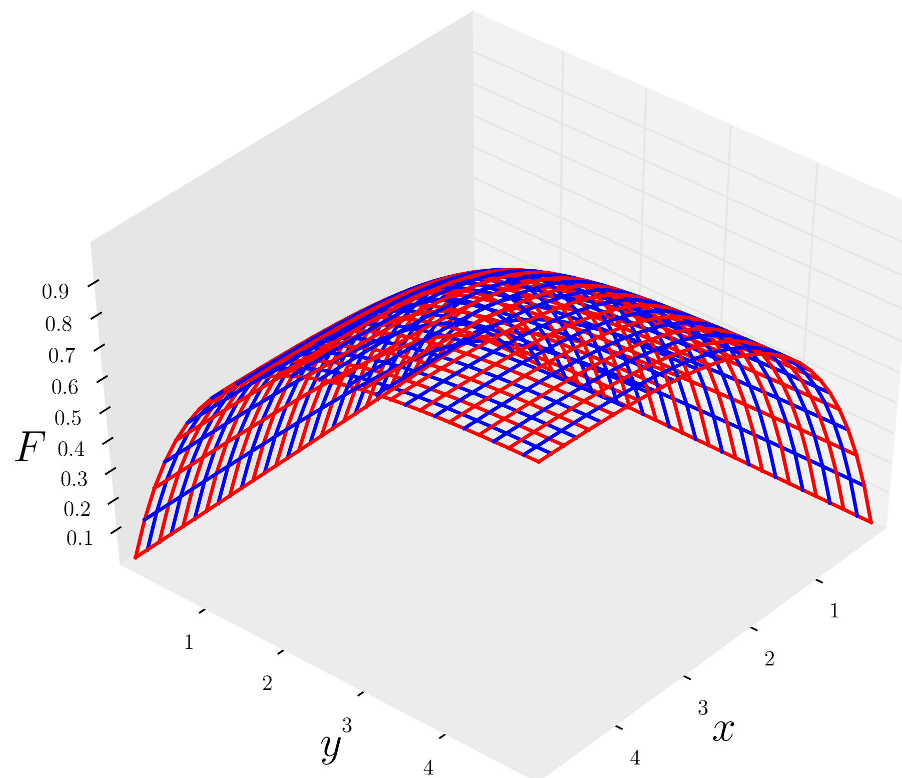
Let $F_{X,Y}(x,y) \;=\; \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{for} \;\; x \geq 0 \;\; \text{and} \;\; y \geq 0 \, , \\ 0 & \text{otherwise} \;\; . \end{cases}$

- Verify that

$$f_{X,Y}(x,y) \;=\; \frac{\partial^2 F}{\partial x \partial y} \;=\; \begin{cases} e^{-x-y} & \text{for} \;\; x \geq 0 \;\; \text{and} \;\; y \geq 0 \, , \\ 0 & \text{otherwise} \;\; . \end{cases}$$



Density function $f_{X,Y}(x,y)$



Distribution function $F_{X,Y}(x,y)$

EXERCISE : (continued $\cdots$)

$$F_{X,Y}(x,y) \;=\; (1-e^{-x})(1-e^{-y}) \quad , \quad f_{X,Y}(x,y) \;=\; e^{-x-y} \; , \quad \text{for } x,y \geq 0 \; .$$

Also verify the following :

- $F(0,0) = 0 \quad , \quad F(\infty,\infty) = 1 \; ,$

- $\int_0^\infty \int_0^\infty \; f_{X,Y}(x,y) \; dx \; dy \;=\; 1 \; ,$ (Why *zero* lower limits ?)

- $f_X(x) \;=\; \int_0^\infty e^{-x-y} \; dy \;=\; e^{-x} \; ,$

- $f_Y(y) \;=\; \int_0^\infty e^{-x-y} \; dx \;=\; e^{-y} \; .$

- $f_{X,Y}(x,y) \;=\; f_X(x) \; \cdot \; f_Y(y) \; .$ (So ?)

EXERCISE : (continued $\cdots$)

$$F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y}) \quad , \quad f_{X,Y}(x,y) = e^{-x-y} \, , \quad \text{for } x, y \geq 0 \, .$$

Also verify the following :

- $F_X(x) = \int_0^x f_X(x)\, dx = \int_0^x e^{-x}\, dx = 1 - e^{-x}$,
- $F_Y(y) = \int_0^y f_Y(y)\, dy = \int_0^y e^{-y}\, dy = 1 - e^{-y}$,
- $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$. (So ?)
- $P(1 < x < \infty) = F_X(\infty) - F_X(1) = 1-(1-e^{-1}) = e^{-1} \cong 0.37$,
- $P(\, 1 < x \leq 2 \, , \; 0 < y \leq 1 \,) = \int_0^1 \int_1^2 e^{-x-y}\, dx\, dy$

$$= (e^{-1} - e^{-2})(1 - e^{-1}) \cong 0.15 \, ,$$

Independent continuous random variables

Recall that two events E and F are *independent* if

$$P(EF) \;=\; P(E)\,P(F)\;.$$

Continuous random variables $X(s)$ and $Y(s)$ are *independent* if

$$P(X \in I_X\;,\;Y \in I_Y) \;=\; P(X \in I_X) \cdot P(Y \in I_Y)\;,$$

for *all* allowable sets I_X and I_Y (typically *intervals*) of *real numbers*.

Equivalently, $X(s)$ and $Y(s)$ are independent if for all such sets I_X and I_Y the *events*

$$X^{-1}(I_X) \quad \text{and} \quad Y^{-1}(I_Y)\;,$$

are independent *in the sample space* $\mathcal{S}$.

NOTE :

$$X^{-1}(I_X) \equiv \{s \in \mathcal{S} \;:\; X(s) \in I_X\}\;,$$
$$Y^{-1}(I_Y) \equiv \{s \in \mathcal{S} \;:\; Y(s) \in I_Y\}\;.$$

FACT : $X(s)$ and $Y(s)$ are *independent* if for all x and y

$$f_{X,Y}(x,y) \quad = \quad f_X(x) \ \cdot \ f_Y(y) \ .$$

EXAMPLE : The random variables with density function

$$f_{X,Y}(x,y) \ = \ \begin{cases} e^{-x-y} & \text{for } x \geq 0 \ \text{ and } \ y \geq 0 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

are *independent* because (by the preceding exercise)

$$f_{X,Y}(x,y) \quad = \quad e^{-x-y} \quad = \quad e^{-x} \cdot e^{-y} \quad = \quad f_X(x) \cdot f_Y(y) \ .$$

NOTE :

$$F_{X,Y}(x,y) \ = \ \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{for } x \geq 0 \ \text{ and } \ y \geq 0 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

also satisfies (by the preceding exercise)

$$F_{X,Y}(x,y) \quad = \quad F_X(x) \cdot F_Y(y) \ .$$

PROPERTY :

For *independent* continuous random variables X and Y we have

$$F_{X,Y}(x,y) \quad = \quad F_X(x) \cdot F_Y(y) \ , \qquad \text{for all } x, y \ .$$

PROOF :

$$\begin{aligned}
F_{X,Y}(x,y) \quad &= \quad P(X \leq x \ , \ Y \leq y) \\
&= \quad \int_{-\infty}^{x} \ \int_{-\infty}^{y} \ f_{X,Y}(x,y) \ dy \ dx \\
&= \quad \int_{-\infty}^{x} \ \int_{-\infty}^{y} \ f_X(x) \cdot f_Y(y) \ dy \ dx \qquad \text{(by independence)} \\
&= \quad \int_{-\infty}^{x} \ [\ f_X(x) \cdot \int_{-\infty}^{y} \ f_Y(y) \ dy \ ] \ dx \\
&= \quad [\ \int_{-\infty}^{x} \ f_X(x) \ dx \ ] \ \cdot \ [\ \int_{-\infty}^{y} \ f_Y(y) \ dy \ ] \\
&= \quad F_X(x) \cdot F_Y(y) \ .
\end{aligned}$$

REMARK : Note how the proof parallels that for the discrete case !

Conditional distributions

Let X and Y be continuous random variables.

For given allowable sets I_X and I_Y (typically *intervals*), let

$$E_x \;=\; X^{-1}(I_X) \qquad \text{and} \qquad E_y \;=\; Y^{-1}(I_Y) \;,$$

be their corresponding *events* in the sample space $\mathcal{S}$.

We have
$$P(E_x|E_y) \;\equiv\; \frac{P(E_xE_y)}{P(E_y)} \;.$$

The *conditional probability density function* is defined as

$$f_{X|Y}(x|y) \;\equiv\; \frac{f_{X,Y}(x,y)}{f_Y(y)} \;.$$

When X and Y are *independent* then

$$f_{X|Y}(x|y) \;\equiv\; \frac{f_{X,Y}(x,y)}{f_Y(y)} \;=\; \frac{f_X(x)\; f_Y(y)}{f_Y(y)} \;=\; f_X(x) \;,$$

(assuming $f_Y(y) \neq 0$).

EXAMPLE : The random variables with density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x \geq 0 \text{ and } y \geq 0 \ , \\ \\ 0 & \text{otherwise} \ , \end{cases}$$

have (by previous exercise) the marginal density functions

$$f_X(x) = e^{-x} \quad , \quad f_Y(y) = e^{-y} \ ,$$

for $x \geq 0$ and $y \geq 0$, and zero otherwise.

Thus for such x, y we have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-x-y}}{e^{-y}} = e^{-x} = f_X(x) \ ,$$

i.e., information about Y does not alter the density function of X .

Indeed, we have already seen that X and Y are *independent* .

Expectation

The *expected value* of a continuous random variable X is

$$E[X] \quad = \quad \int_{-\infty}^{\infty} x \ f_X(x) \ dx \ ,$$

which represents the *average value* of X over many trials.

The expected value of a *function of a random variable* is

$$E[g(X)] \quad = \quad \int_{-\infty}^{\infty} g(x) \ f_X(x) \ dx \ .$$

The expected value of a function of *two* random variables is

$$E[g(X,Y)] \quad = \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \ f_{X,Y}(x,y) \ dy \ dx \ .$$

EXAMPLE :

For the *pointer* experiment

$$f_X(x) \quad = \quad \begin{cases} 0 \ , & x \leq 0 \\ \\ 1 \ , & 0 < x \leq 1 \\ \\ 0 \ , & 1 < x \end{cases}$$

we have

$$E[X] \quad = \quad \int_{-\infty}^{\infty} x \ f_X(x) \ dx \quad = \quad \int_0^1 x \ dx \quad = \quad \frac{x^2}{2}\Big|_0^1 \quad = \quad \frac{1}{2} \ ,$$

and

$$E[X^2] \quad = \quad \int_{-\infty}^{\infty} x^2 \ f_X(x) \ dx \quad = \quad \int_0^1 x^2 \ dx \quad = \quad \frac{x^3}{3}\Big|_0^1 \quad = \quad \frac{1}{3} \ .$$

EXAMPLE : For the joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0 \ , \\ \\ 0 & \text{otherwise} \ . \end{cases}$$

we have (by previous exercise) the marginal density functions

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x > 0 \ , \\ \\ 0 & \text{otherwise} \ , \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} e^{-y} & \text{for } y > 0 \ , \\ \\ 0 & \text{otherwise} \ . \end{cases}$$

Thus $\quad E[X] \ = \ \int_0^\infty x \, e^{-x} \, dx \ = \ -[(x+1)e^{-x}] \Big|_0^\infty \ = \ 1 \ . \quad$ (Check !)

Similarly $\qquad E[Y] \ = \ \int_0^\infty y \, e^{-y} \, dy \ = \ 1 \ ,$

and

$$E[XY] \ = \ \int_0^\infty \int_0^\infty xy \, e^{-x-y} \, dy \, dx \ = \ 1 \ . \qquad (\text{ Check ! })$$

EXERCISE :

Prove the following for *continuous* random variables :

- $E[aX] \quad = \quad a\ E[X]$,

- $E[aX + b] \quad = \quad a\ E[X] \quad + \quad b$,

- $E[X + Y] \quad = \quad E[X] \quad + \quad E[Y]$,

and *compare* the proofs to those for *discrete* random variables.

EXERCISE :

A stick of length 1 is split at a randomly selected point X.
(Thus X is uniformly distributed in the interval $[0, 1]$.)
Determine the expected length of the piece containing the point $1/3$.

PROPERTY : If X and Y are *independent* then

$$E[XY] \quad = \quad E[X] \cdot E[Y] \, .$$

PROOF :

$$\begin{aligned} E[XY] \quad &= \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \; x \; y \; f_{X,Y}(x,y) \;\; dy \; dx \\ &= \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \; x \; y \; f_X(x) \; f_Y(y) \;\; dy \; dx \qquad \text{(by independence)} \\ &= \quad \int_{\mathbb{R}} [\; x \; f_X(x) \;\; \int_{\mathbb{R}} \; y \; f_Y(y) \; dy \;] \; dx \\ &= \quad [\int_{\mathbb{R}} \; x \; f_X(x) \; dx \;] \; \cdot \; [\int_{\mathbb{R}} \; y \; f_Y(y) \; dy \;] \\ &= \quad E[X] \cdot E[Y] \, . \end{aligned}$$

REMARK : Note how the proof parallels that for the discrete case !

EXAMPLE : For

$$f_{X,Y}(x,y) \;=\; \begin{cases} e^{-x-y} & \text{for } x>0 \text{ and } y>0 \;, \\ \\ 0 & \text{otherwise} \;, \end{cases}$$

we already found

$$f_X(x) \;=\; e^{-x} \qquad , \qquad f_Y(y) \;=\; e^{-y} \;,$$

so that

$$f_{X,Y}(x,y) \;=\; f_X(x) \cdot f_Y(y) \;,$$

i.e., X and Y are *independent* .

Indeed, we also already found that

$$E[X] \;=\; E[Y] \;=\; E[XY] \;=\; 1 \;,$$

so that

$$E[XY] \;=\; E[X] \cdot E[Y] \;.$$

Variance

Let $\mu \;=\; E[X] \;=\; \int_{-\infty}^{\infty} x\, f_X(x)\, dx$

Then the *variance* of the continuous random variable X is

$$Var(X) \;\equiv\; E[\,(X-\mu)^2] \;\equiv\; \int_{-\infty}^{\infty} (x-\mu)^2\, f_X(x)\, dx\ ,$$

which is the average weighted *square distance* from the mean.

As in the discrete case, we have

$$\begin{aligned} Var(X) &= E[X^2 - 2\mu X \;+\; \mu^2] \\ &= E[X^2] - 2\mu E[X] \;+\; \mu^2 \qquad = \;E[X^2] \;-\; \mu^2\ . \end{aligned}$$

The *standard deviation* of X is

$$\sigma(X) \;\equiv\; \sqrt{Var(X)} \;=\; \sqrt{E[X^2] \;-\; \mu^2}\ .$$

which is the average weighted *distance* from the mean.

EXAMPLE : For $f(x) = \begin{cases} e^{-x} , & x > 0 , \\ 0 , & x \leq 0 , \end{cases}$

we have

$$E[X] = \mu = \int_0^\infty x\, e^{-x}\, dx = 1 \qquad (\text{ already done ! }) ,$$

$$E[X^2] = \int_0^\infty x^2\, e^{-x}\, dx = -\left[(x^2 + 2x + 2)e^{-x}\right]\Big|_0^\infty = 2 ,$$

$$Var(X) = E[X^2] - \mu^2 = 2 - 1^2 = 1 ,$$

$$\sigma(X) = \sqrt{Var(X)} = 1 .$$

NOTE : The two integrals can be done by "*integration by parts* ".

EXERCISE :

Also use the *Method of Moments* to compute $E[X]$ and $E[X^2]$.

EXERCISE : For the random variable X with density function

$$f(x) \quad = \quad \begin{cases} 0\ , & x \leq -1 \\ \mathrm{c}\ , & -1 < x \leq 1 \\ 0\ , & x > 1 \end{cases}$$

- Determine the value of c
- Draw the graph of $f(x)$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \leq -\frac{1}{2})$
- Determine $P(\mid X \mid \geq \frac{1}{2})$

EXERCISE : For the random variable X with density function

$$f(x) \quad = \quad \begin{cases} x+1\ , & -1 < x \leq 0 \\ 1-x\ , & 0 < x \leq 1 \\ 0\ , & \text{otherwise} \end{cases}$$

- Draw the graph of $f(x)$
- Verify that $\int_{-\infty}^{\infty} f(x)\ dx = 1$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \geq \frac{1}{3})$
- Determine $P(\mid X \mid \leq \frac{1}{3})$

EXERCISE : For the random variable X with density function

$$f(x) \quad = \quad \begin{cases} \frac{3}{4}\,(1-x^2)\ , & -1 < x \leq 1 \\ 0\ , & \text{otherwise} \end{cases}$$

- Draw the graph of $f(x)$
- Verify that $\int_{-\infty}^{\infty} f(x)\ dx = 1$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \leq 0)$
- Compute $P(X \geq \frac{2}{3})$
- Compute $P(\mid X \mid \geq \frac{2}{3})$

EXERCISE : Recall the density function

$$f_n(x) = \begin{cases} cx^n(1-x^n)\ , & 0 \le x \le 1 \\ \\ 0\ , & \text{otherwise} \end{cases}$$

considered earlier, where n is a positive integer, and where

$$c \;=\; \tfrac{(n+1)(2n+1)}{n}\ .$$

- Determine $E[X]$.

- What happens to $E[X]$ for *large* n ?

- Determine $E[X^2]$

- What happens to $E[X^2]$ for *large* n ?

- What happens to $Var(X)$ for *large* n ?

Covariance

Let X and Y be continuous random variables with *mean*

$$E[X] \;=\; \mu_X \quad , \quad E[Y] \;=\; \mu_Y \; .$$

Then the *covariance* of X and Y is

$$Cov(X,Y) \;\equiv\; E[\,(X-\mu_X)\,(Y-\mu_Y)\,]$$

$$=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)\,(y-\mu_Y)\,f_{X,Y}(x,y)\;dy\;dx \; .$$

As in the discrete case, we have

$$Cov(X,Y) \;=\; E[\,(X-\mu_X)\,(Y-\mu_Y)\,]$$

$$=\; E[XY-\mu_X Y-\mu_Y X \;+\; \mu_X\mu_Y]$$

$$=\; E[XY] \;-\; E[X]\;E[Y] \; .$$

As in the discrete case, we also have

PROPERTY 1 :

- $Var(X+Y) \; = \; Var(X) \, + \, Var(Y) \, + \, 2\,Cov(X,Y)$,

and

PROPERTY 2 : If X and Y are *independent* then

- $Cov(X,Y) \; = \; 0$,

- $Var(X+Y) \; = \; Var(X) \, + \, Var(Y)$.

NOTE :

- The proofs are identical to those for the discrete case !

- As in the discrete case, if $Cov(X,Y) \; = \; 0$ then X and Y are not necessarily independent!

EXAMPLE : For

$$f_{X,Y}(x,y) \;=\; \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0 \,, \\ \\ 0 & \text{otherwise} \,, \end{cases}$$

we already found

$$f_X(x) \;=\; e^{-x} \qquad , \qquad f_Y(y) \;=\; e^{-y} \;,$$

so that

$$f_{X,Y}(x,y) \;=\; f_X(x) \cdot f_Y(y) \;,$$

i.e., X and Y are *independent* .

Indeed, we also already found

$$E[X] \;=\; E[Y] \;=\; E[XY] \;=\; 1 \;,$$

so that

$$Cov(X,Y) \;=\; E[XY] \;-\; E[X]\,E[Y] \;=\; 0 \;.$$

EXERCISE :

Verify the following properties :

- $$Var(cX + d) \;=\; c^2\, Var(X) \;,$$

- $$Cov(X,Y) \;=\; Cov(Y,X) \;,$$

- $$Cov(cX,Y) \;=\; c\, Cov(X,Y) \;,$$

- $$Cov(X,cY) \;=\; c\, Cov(X,Y) \;,$$

- $$Cov(X+Y,Z) \;=\; Cov(X,Z) \;+\; Cov(Y,Z) \;,$$

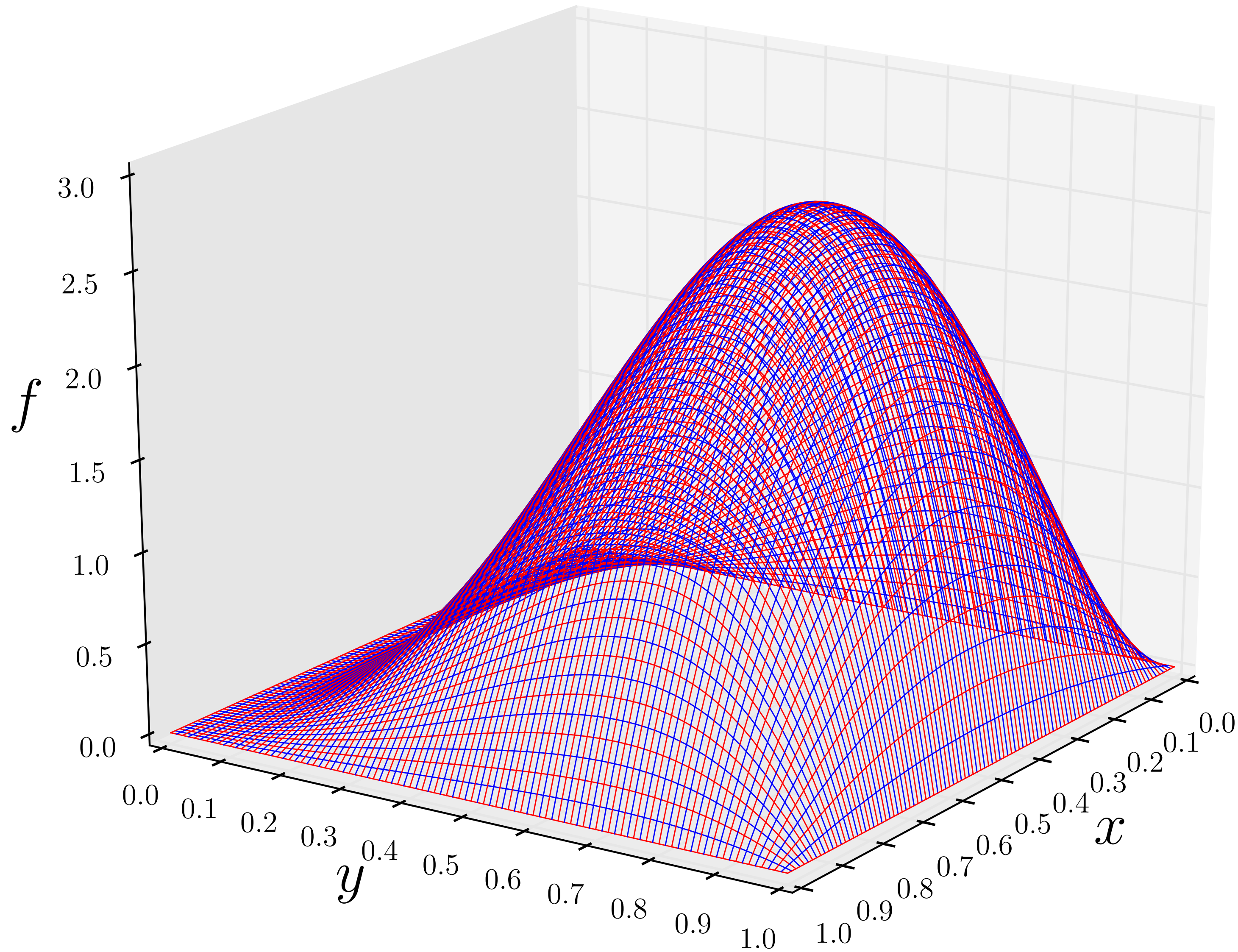
- $$Var(X+Y) \;=\; Var(X) \;+\; Var(Y) \;+\; 2\, Cov(X,Y) \;.$$

EXERCISE :

For the random variables X , Y with *joint density function*

$$f(x,y) = \begin{cases} 45xy^2(1-x)(1-y^2)\ , & 0 \le x \le 1\ ,\ 0 \le y \le 1 \\ 0\ , & \text{otherwise} \end{cases}$$

- Verify that $\int_0^1 \int_0^1 f(x,y)\ dy\ dx = 1$.
- Determine the *marginal density functions* $f_X(x)$ and $f_Y(y)$.
- Are X and Y *independent* ?
- What is the value of $Cov(X,Y)$?



The joint probability density function $f_{XY}(x, y)$.

Markov's inequality.

For a continuous *nonnegative* random variable X , and $c > 0$, we have

$$P(X \geq c) \quad \leq \quad \frac{E[X]}{c} \ .$$

PROOF :

$$\begin{aligned} E[X] \quad &= \quad \int_0^\infty x f(x)\ dx \quad = \quad \int_0^c x f(x)\ dx \quad + \quad \int_c^\infty x f(x)\ dx \\ &\geq \quad \int_c^\infty x f(x)\ dx \\ &\geq \quad c \int_c^\infty f(x)\ dx \qquad\qquad (\text{ Why ? }) \\ &= \quad c\ P(X \geq c)\ . \end{aligned}$$

EXERCISE :

Show Markov's inequality also holds for *discrete* random variables.

Markov's inequality : For continuous *nonnegative* X , $c > 0$:

$$P(X \geq c) \quad \leq \quad \frac{E[X]}{c} \ .$$

EXAMPLE : For

$$f(x) \ = \ \begin{cases} e^{-x} & \text{for } x > 0 \ , \\ \\ 0 & \text{otherwise} \ , \end{cases}$$

we have

$$E[X] \quad = \quad \int_0^\infty x \ e^{-x} \ dx \quad = \quad 1 \qquad\qquad (\text{ already done ! })$$

Markov's inequality gives

$$c = \mathbf{1} \ : \qquad P(X \geq \mathbf{1}) \quad \leq \quad \frac{E[X]}{\mathbf{1}} \quad = \quad \frac{1}{\mathbf{1}} \quad = \quad 1 \ \ (\ ! \ )$$

$$c = \mathbf{10} \ : \qquad P(X \geq \mathbf{10}) \quad \leq \quad \frac{E[X]}{\mathbf{10}} \quad = \quad \frac{1}{10} \quad = \quad 0.1$$

QUESTION : Are these estimates " *sharp* " ?

QUESTION : Are these estimates "*sharp*" ?

Markov's inequality gives

$$c = 1 \ : \qquad P(X \geq 1) \ \leq \ \frac{E[X]}{1} \ = \ \frac{1}{1} \ = \ 1 \ \ (\ !\ )$$

$$c = 10 \ : \qquad P(X \geq 10) \ \leq \ \frac{E[X]}{10} \ = \ \frac{1}{10} \ = \ 0.1$$

The actual values are

$$P(X \geq 1) \ = \ \int_1^\infty e^{-x}\, dx \ = \ e^{-1} \ \cong \ 0.37$$

$$P(X \geq 10) \ = \ \int_{10}^\infty e^{-x}\, dx \ = \ e^{-10} \ \cong \ 0.000045$$

EXERCISE : Suppose the score of students taking an examination is a random variable with mean 65 .

Give an upper bound on the probability that a student's score is greater than 75 .

Chebyshev's inequality: For (practically) any random variable X:

$$P(\ |X-\mu|\ \geq\ k\ \sigma\ )\ \leq\ \frac{1}{k^2}\ ,$$

where $\mu = E[X]$ is the *mean*, $\sigma = \sqrt{Var(X)}$ the *standard deviation*.

PROOF : Let $Y \equiv (X-\mu)^2$, which is nonnegative.

By Markov's inequality

$$P(Y \geq c)\ \leq\ \frac{E[Y]}{c}\ .$$

Taking $c = k^2\sigma^2$ we have

$$P(\ |X-\mu|\ \geq\ k\sigma\ )\ =\ P(\ (X-\mu)^2\ \geq\ k^2\sigma^2\ )\ =\ P(\ Y\ \geq\ k^2\sigma^2\ )$$

$$\leq\ \frac{E[\ Y\ ]}{k^2\sigma^2}\ =\ \frac{Var(X)}{k^2\sigma^2}\ =\ \frac{\sigma^2}{k^2\sigma^2}\ =\ \frac{1}{k^2}\ .\quad \textbf{QED}\ !$$

NOTE : This inequality also holds for discrete random variables.

EXAMPLE : Suppose the value of the Canadian dollar in terms of the US dollar over a certain period is a random variable X with

$$\textit{mean} \quad \mu = 0.98 \quad \text{and} \quad \textit{standard deviation} \quad \sigma = 0.05 \ .$$

What can be said of the probability that the Canadian dollar is valued

$$\text{between} \quad \$0.88\text{US} \quad \text{and} \quad \$1.08\text{US} \ ,$$

that is,

$$\text{between} \quad \mu - 2\sigma \quad \text{and} \quad \mu + 2\sigma \ ?$$

SOLUTION : By Chebyshev's inequality we have

$$P(\ | X - \mu | \ \geq \ 2 \, \sigma \ ) \ \leq \ \frac{1}{2^2} \ = \ 0.25 \ .$$

Thus

$$P(\ | X - \mu | \ < \ 2 \, \sigma \ ) \ > \ 1 \ - \ 0.25 \ = \ 0.75 \ ,$$

that is,

$$P(\ \$0.88\text{US} \ < \ \text{Can}\$ \ < \ \$1.08\text{US} \ ) \ > \ 75 \ \% \ .$$

EXERCISE :

The score of students taking an examination is a random variable with mean $\mu = 65$ and standard deviation $\sigma = 5$.

- What is the probability a student scores between 55 and 75 ?
- How many students would have to take the examination so that the probability that their average grade is between 60 and 70 is at least 80% ?

  HINT : Defining

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) , \quad (\text{ the average grade })$$

we have

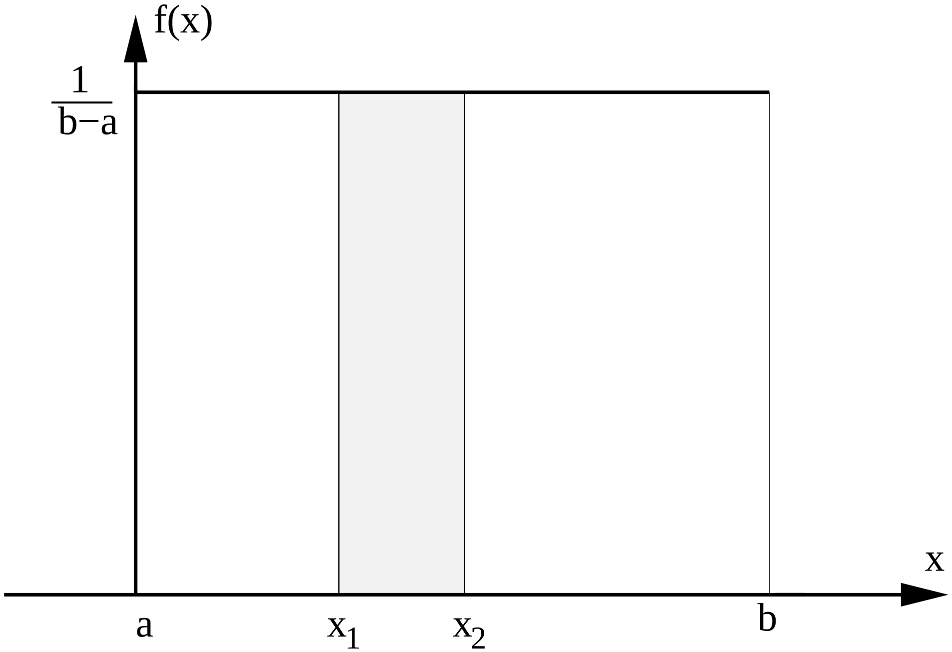
$$\mu_{\bar{X}} = E[\bar{X}] = \frac{1}{n}\, n\, \mu = \mu = 65 ,$$

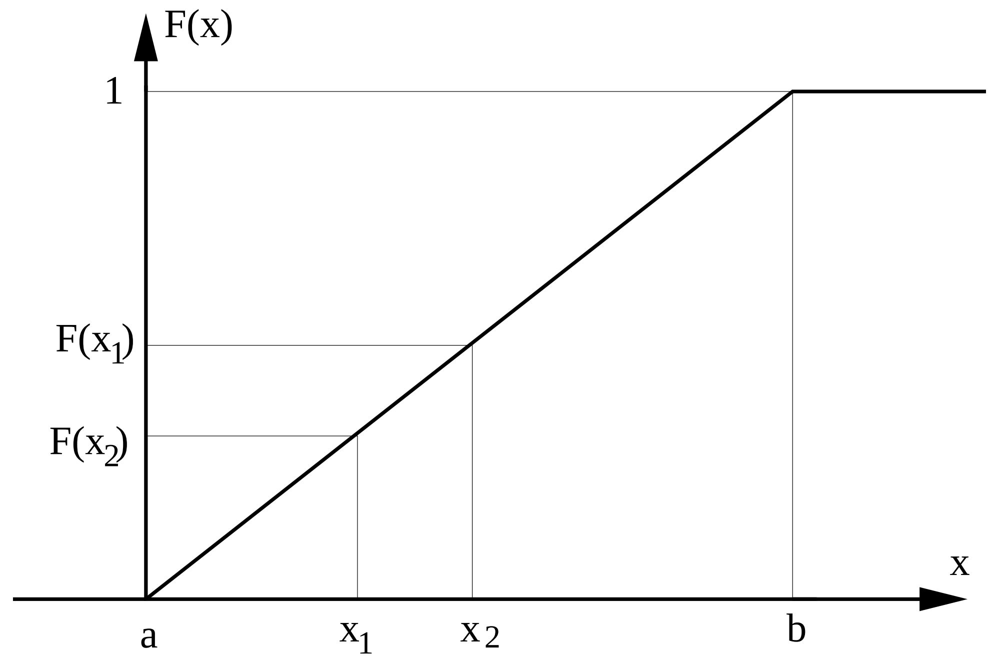
and, assuming independence,

$$Var(\bar{X}) = n\, \frac{\sigma^2}{n^2} = \frac{\sigma^2}{n} = \frac{25}{n} , \quad \text{and} \quad \sigma_{\bar{X}} = \frac{5}{\sqrt{n}} .$$

The Uniform Random Variable

$$f(x) = \begin{cases} \frac{1}{b-a}\ , & a < x \leq b \\ \\ 0\ , & \text{otherwise} \end{cases}\ , \qquad F(x) = \begin{cases} 0\ , & x \leq a \\ \frac{x-a}{b-a}\ , & a < x \leq b \\ 1\ , & x > b \end{cases}$$





(Already introduced earlier for the special case $a = 0$, $b = 1$.)

EXERCISE :

Show that the *uniform density function*

$$f(x) = \begin{cases} \frac{1}{b-a} \, , & a < x \leq b \\ \\ 0 \, , & \text{otherwise} \end{cases}$$

has *mean*

$$\mu \quad = \quad \frac{a+b}{2} \, ,$$

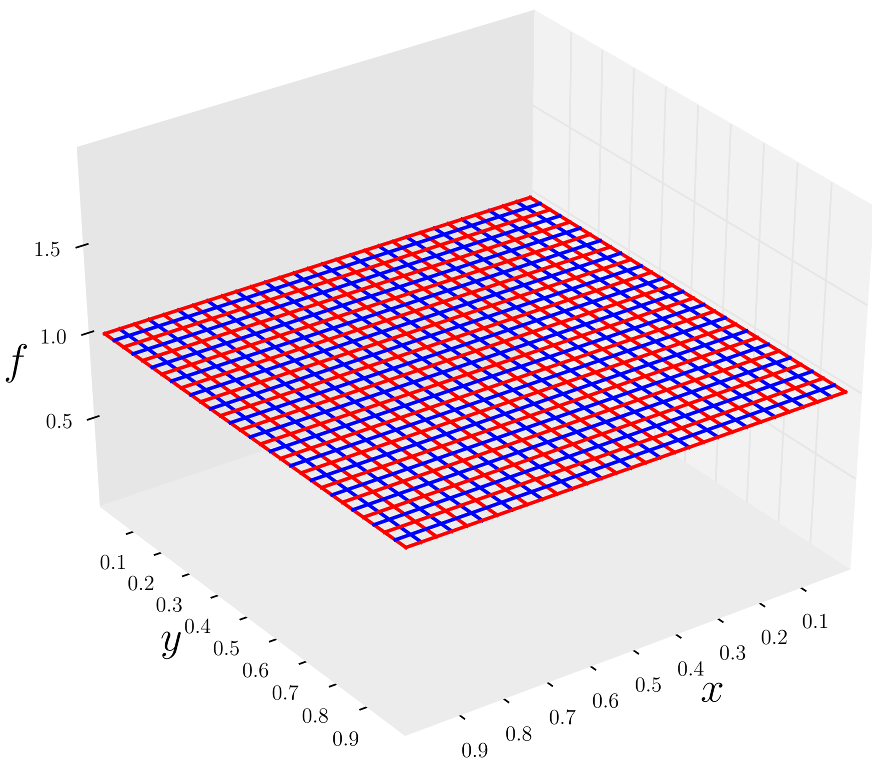
and *standard deviation*

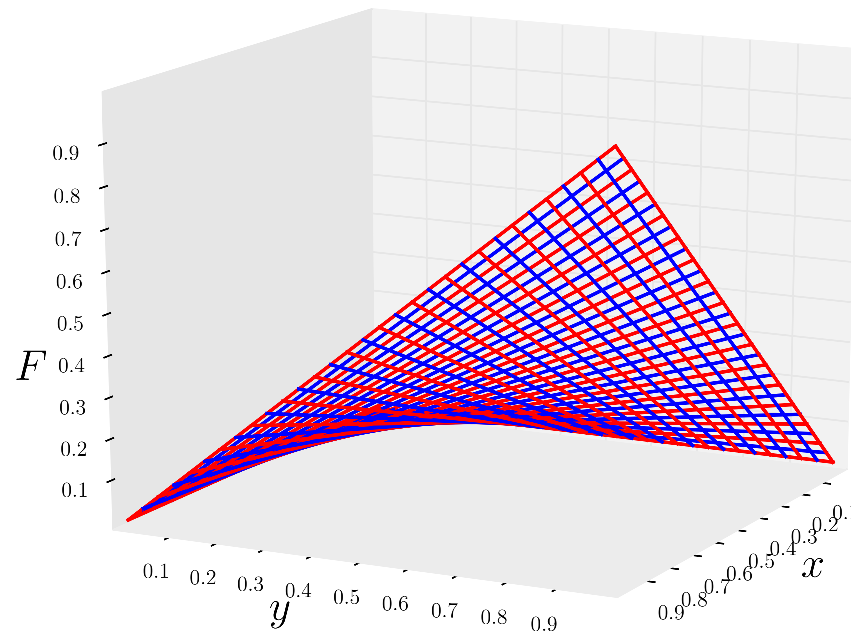
$$\sigma \quad = \quad \frac{b-a}{2\sqrt{3}} \, .$$

A *joint uniform* random variable :

$$f(x, y) \;=\; \frac{1}{(b-a)(d-c)} \quad , \quad F(x, y) \;=\; \frac{(x-a)(y-c)}{(b-a)(d-c)} \; ,$$

for $x \in (a, b]$, $y \in (c, d]$.





Here $x \in [0, 1]$, $y \in [0, 1]$.

EXERCISE :

Consider the *joint uniform density function*

$$f(x,y) \quad = \quad \begin{cases} c & \text{for} \quad x^2+y^2 \ \leq \ 4 \ , \\ \\ 0 & \text{otherwise} \ . \end{cases}$$

- What is the value of c ?
- What is $P(X<0)$?
- What is $P(X<0 \ , \ Y<0)$?
- What is $f(\ x \mid y=1 \ )$?

HINT : No complicated calculations are needed !

The Exponential Random Variable

$$f(x) \ = \ \begin{cases} \lambda e^{-\lambda x} \ , & x > 0 \\ \\ 0 \ , & x \leq 0 \end{cases} \quad , \quad F(x) \ = \ \begin{cases} 1 - e^{-\lambda x} \ , & x > 0 \\ \\ 0 \ , & x \leq 0 \end{cases}$$

with

$$E[X] \ = \ \mu \ = \ \int_0^\infty x \ \lambda e^{-\lambda x} \ dx \ = \ \tfrac{1}{\lambda} \qquad (\text{ Check ! }) \ ,$$

$$E[X^2] \ = \ \int_0^\infty x^2 \ \lambda e^{-\lambda x} \ dx \ = \ \tfrac{2}{\lambda^2} \qquad (\text{ Check ! }) \ ,$$

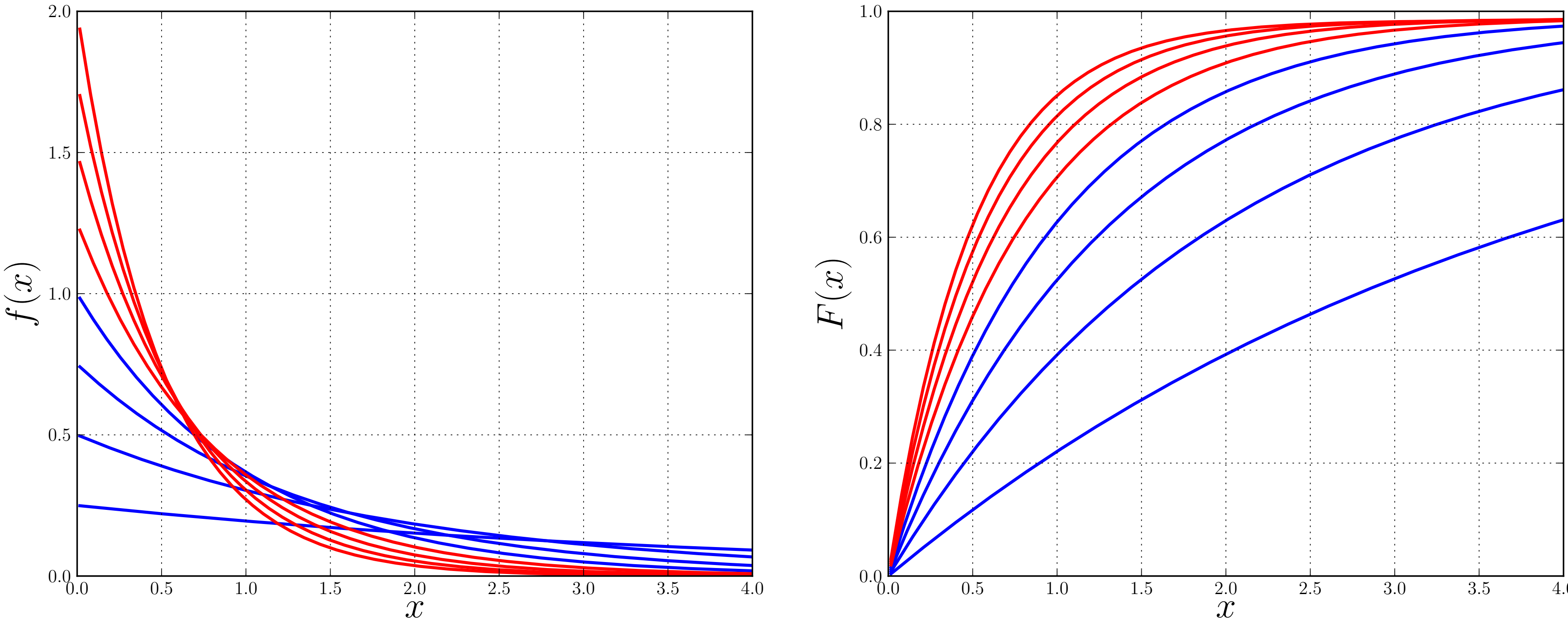
$$Var(X) \ = \ E[X^2] \ - \ \mu^2 \ = \ \tfrac{1}{\lambda^2} \ ,$$

$$\sigma(X) \ = \ \sqrt{Var(X)} \ = \ \tfrac{1}{\lambda} \ .$$

NOTE : The two integrals can be done by "*integration by parts*".

EXERCISE : (Done earlier for $\lambda = 1$) :

Also use the *Method of Moments* to compute $E[X]$ and $E[X^2]$.



The Exponential *density* and *distribution* functions

$$f(x) \;=\; \lambda e^{-\lambda x} \qquad , \qquad F(x) \;=\; 1 - e^{-\lambda x} \; ,$$

for $\lambda = 0.25, 0.50, 0.75, 1.00$ (*blue*), $1.25, 1.50, 1.75, 2.00$ (*red*).

PROPERTY : From

$$F(x) \quad \equiv \quad P(X \le x) \quad = \quad 1 - e^{-\lambda x} \ , \qquad\qquad (\text{ for } \ x > 0 \ ) \ ,$$

we have
$$P(X > x) \quad = \quad 1 \ - \ (1 - e^{-\lambda x}) \quad = \quad e^{-\lambda x} \ .$$

Also, for $\Delta x > 0$,

$$P(X > x + \Delta x \mid X > x) \ = \ \frac{P(X > x + \Delta x \ , \ X > x)}{P(X > x)}$$

$$= \ \frac{P(X > x + \Delta x)}{P(X > x)} \ = \ \frac{e^{-\lambda(x+\Delta x)}}{e^{-\lambda x}} \ = \ e^{-\lambda \Delta x} \ .$$

CONCLUSION : $\qquad P(\ X > x + \Delta x \ \mid \ X > x \ )$

only depends on Δx (and λ), and *not* on x !

We say that the exponential random variable is "*memoryless*" .

EXAMPLE :

Let the density function $f(t)$ model *failure* of a device,

$$f(t) \quad = \quad e^{-t} \; , \qquad\qquad (\text{taking } \; \lambda = 1 \;) \; ,$$

i.e., the *probability of failure* in the time-interval $(0, t]$ is

$$F(t) \quad = \quad \int_0^t f(t) \; dt \quad = \quad \int_0^t e^{-t} \; dt \quad = \quad 1 \; - \; e^{-t} \; ,$$

with

$$F(0) \; = \; 0 \; , \qquad\qquad (\text{ the device works at time } 0 \text{ }).$$

and

$$F(\infty) \; = \; 1 \; , \qquad\qquad (\text{ the device must fail at some time }).$$

EXAMPLE : (continued $\cdots$) $\quad F(t) \ = \ 1 - e^{-t}$.

Let E_t be the *event* that the device still *works* at time t :

$$P(E_t) \quad = \quad 1 - F(t) \quad = \quad e^{-t} \ .$$

The probability it still works at time $t+1$ is

$$P(E_{t+1}) \quad = \quad 1 - F(t+1) \quad = \quad e^{-(t+1)} \ .$$

The probability it still works at time $t+1$, given it works at time t is

$$P(E_{t+1}|E_t) \quad = \quad \frac{P(E_{t+1}E_t)}{P(E_t)} \quad = \quad \frac{P(E_{t+1})}{P(E_t)} \quad = \quad \frac{e^{-(t+1)}}{e^{-t}} \quad = \quad \frac{1}{e} \ ,$$

which is *independent of* t !

QUESTION : Is such an exponential distribution realistic if the "device" is your heart, and time t is measured in decades ?

The Standard Normal Random Variable

The *standard normal* random variable has *density function*

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \;, \qquad\qquad -\infty < x < \infty \;,$$

with *mean*

$$\mu \;\; = \;\; \int_{-\infty}^{\infty} x\; f(x)\; dx \;\; = \;\; 0 \;, \qquad\qquad (\text{ Check ! })$$

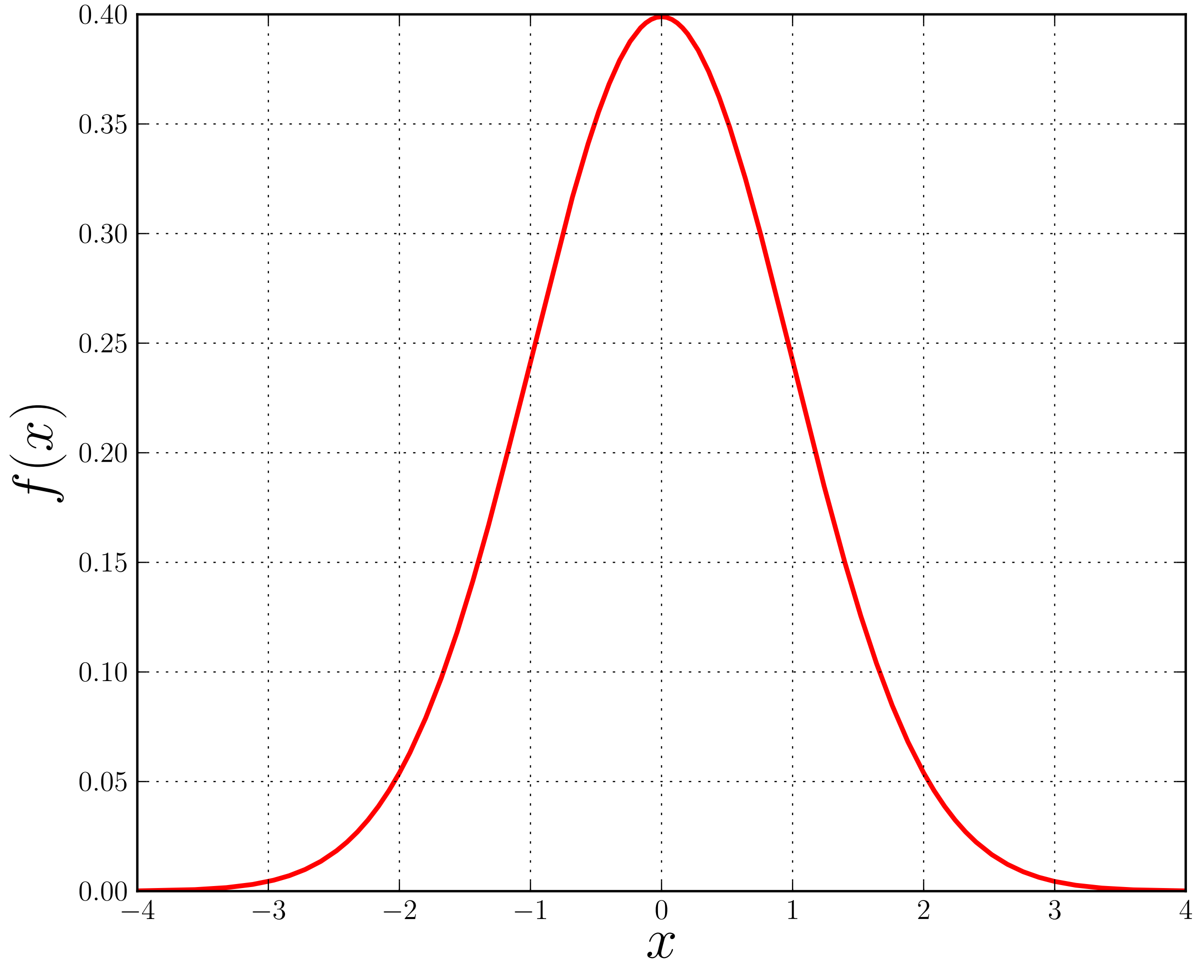
Since

$$E[X^2] \;\; = \;\; \int_{-\infty}^{\infty} x^2\; f(x)\; dx \;\; = \;\; 1 \;, \qquad\qquad (\text{ more difficult } \; \cdots \;)$$

we have

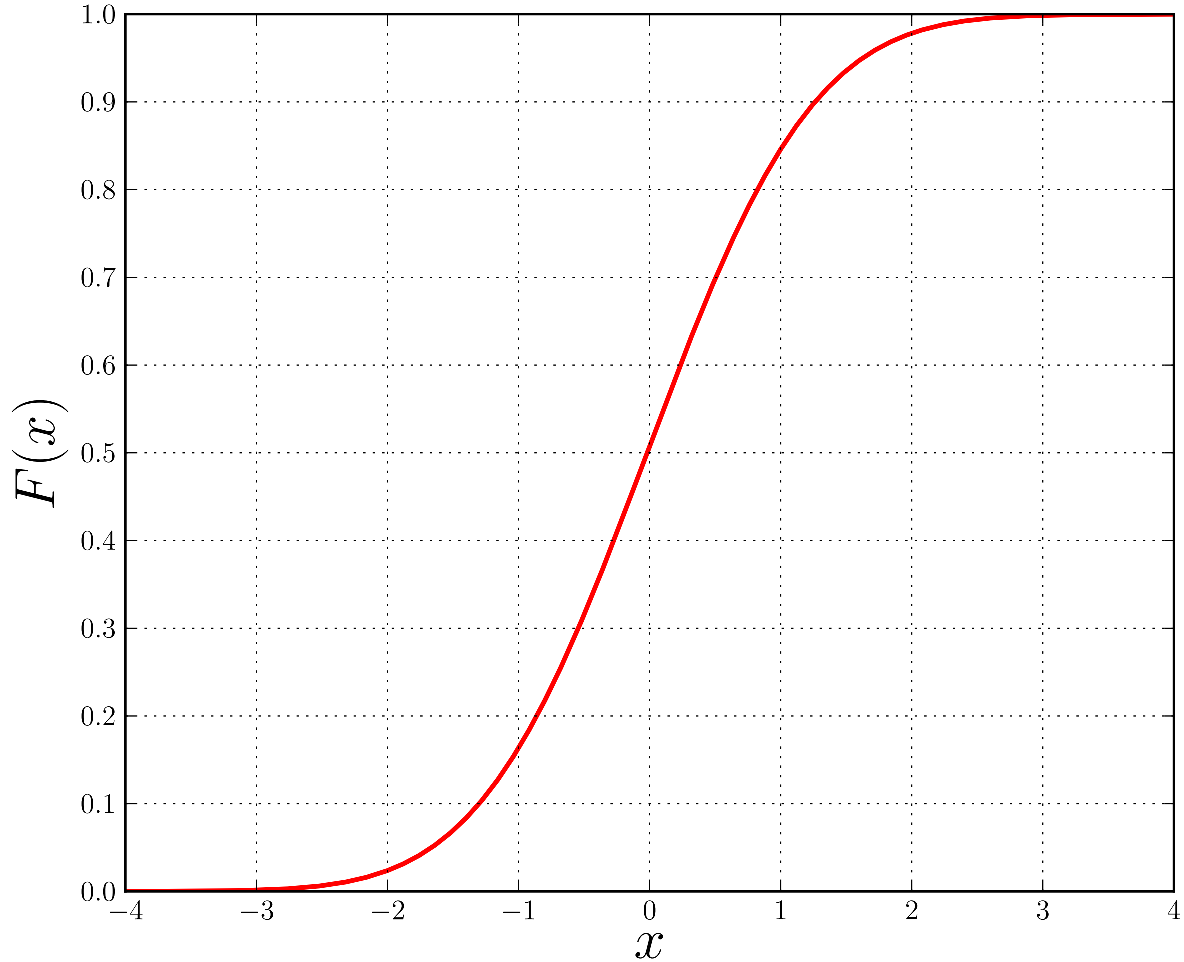
$$Var(X) \;=\; E[X^2] - \mu^2 \;\; = \;\; 1 \;, \qquad \text{and} \qquad \sigma(X) \;=\; 1 \;.$$

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$$



The *standard normal density function* $f(x)$.

$$\boldsymbol{\Phi}(\mathbf{x}) \quad = \quad F(x) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\ dx$$



The *standard normal distribution function* $F(x)$

(often denoted by $\boldsymbol{\Phi}(\mathbf{x})$) .

The Standard Normal Distribution $\Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$
0.0	.5000	-1.2	.1151
-0.1	.4602	-1.4	.0808
-0.2	.4207	-1.6	.0548
-0.3	.3821	-1.8	.0359
-0.4	.3446	**-2.0**	**.0228**
-0.5	.3085	-2.2	.0139
-0.6	.2743	-2.4	.0082
-0.7	.2420	-2.6	.0047
-0.8	.2119	-2.8	.0026
-0.9	.1841	-3.0	.0013
-1.0	.1587	-3.2	.0007

(For example, $P(Z \leq -\mathbf{2.0}) \ = \ \Phi(-\mathbf{2.0}) \ = \ \mathbf{2.28}\%$)

QUESTION : How to get the values of $\Phi(z)$ for *positive* z ?

EXERCISE :

Suppose the random variable X has the *standard normal* distribution.

What are the values of

- $P(\ X \ \leq -0.5\ )$
- $P(\ X \ \leq \ 0.5\ )$
- $P(\ \mid X \mid \ \geq \ 0.5\ )$
- $P(\ \mid X \mid \ \leq \ 0.5\ )$
- $P(\ -1 \ \leq \ X \ \leq \ 1\ )$
- $P(\ -1 \ \leq \ X \ \leq \ 0.5\ )$

The General Normal Random Variable

The *general normal density function* is

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

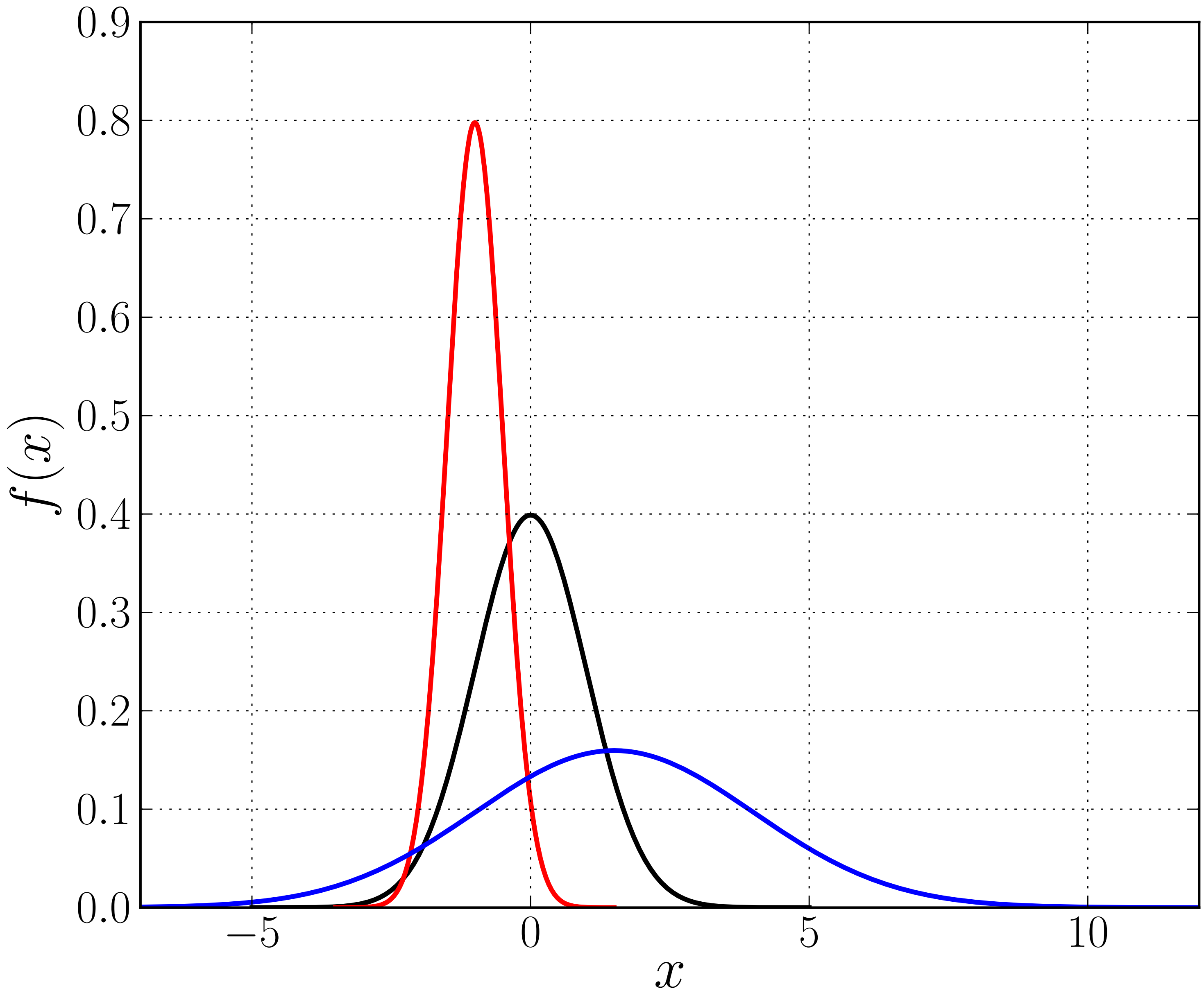
where, not surprisingly,

$$E[X] \quad = \quad \mu \qquad\qquad (\text{ Why ? })$$

One can also show that

$$Var(X) \quad \equiv \quad E[(X-\mu)^2] \quad = \quad \sigma^2 \ ,$$

so that σ is in fact the *standard deviation* .



The standard normal (*black*) and the normal density functions with $\mu = -1$, $\sigma = 0.5$ (*red*) and $\mu = 1.5$, $\sigma = 2.5$ (*blue*).

To compute the *mean* of the *general normal density function*

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

consider

$$E[X-\mu] \;=\; \int_{-\infty}^{\infty}(x-\mu)\ f(x)\ dx$$

$$= \;\; \frac{1}{\sqrt{2\pi}\ \sigma}\ \int_{-\infty}^{\infty}(x-\mu)\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\ \ dx$$

$$= \;\; \frac{-\sigma^2}{\sqrt{2\pi}\ \sigma}\ \ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\ \ \Big|_{-\infty}^{\infty} \quad = \quad 0\ .$$

Thus the *mean* is indeed

$$E[X] \quad = \quad \mu\ .$$

NOTE : If X is *general normal* we have the *very useful formula* :

$$P\left(\frac{X-\mu}{\sigma} \leq c \right) = \Phi(c) \ ,$$

i.e., we can use the *Table* of the *standard normal distribution* !

PROOF : For any constant c we have

$$P\left(\frac{X-\mu}{\sigma} \leq c\right) = P(X \leq \mu + c\sigma) = \frac{1}{\sqrt{2\pi}\ \sigma} \int_{-\infty}^{\mu+c\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} \, dx \ .$$

Let $y \equiv (x-\mu)/\sigma$, so that $x = \mu + y\sigma$.

Then the new variable y ranges from $-\infty$ to c , and

$$(x-\mu)^2/\sigma^2 = y^2 \quad , \quad dx = \sigma \, dy \ ,$$

so that

$$P\left(\frac{X-\mu}{\sigma} \leq c\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{1}{2}y^2} \, dy = \Phi(c) \ .$$

(the *standard normal distribution*)

EXERCISE : Suppose X is normally distributed with

$$\textit{mean} \quad \mu = 1.5 \qquad \text{and} \qquad \textit{standard deviation} \quad \sigma = 2.5 \ .$$

Use the *standard normal Table* to determine :

- $P(X \leq -0.5)$
- $P(X \geq 0.5)$
- $P(\ | X - \mu | \ \geq 0.5)$
- $P(\ | X - \mu | \ \leq 0.5)$

The Chi-Square Random Variable

Suppose $$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ ,$$

are *independent standard normal* random variables.

Then $$\chi_{\mathbf{n}}^{\mathbf{2}} \ \equiv \ X_1^2 + X_2^2 + \cdots + X_n^2 \ ,$$

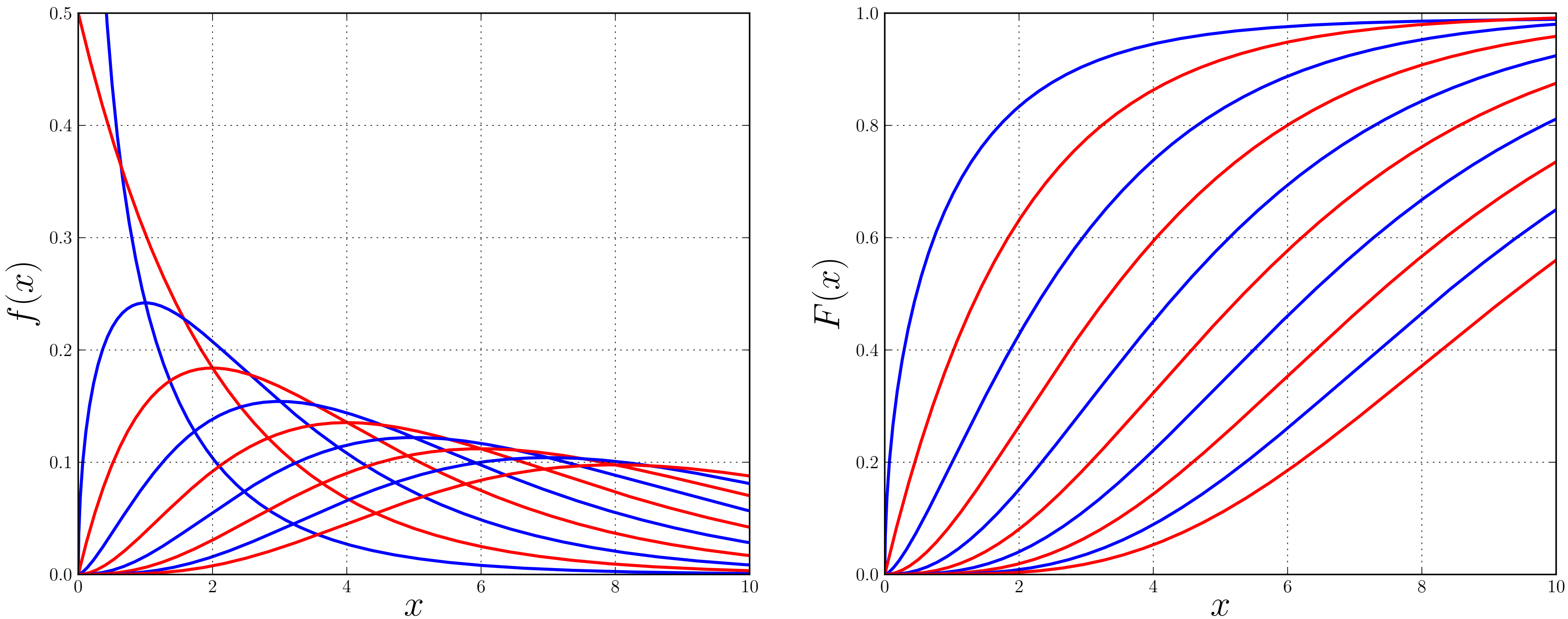
is called the *chi-square random variable* with n *degrees of freedom.*

We will show that

$$E[\chi_n^2] = n \quad , \quad Var(\chi_n^2) = 2n \quad , \quad \sigma(\chi_n^2) = \sqrt{2n} \ .$$

NOTE :

The ^{2} in χ_n^2 is part of its *name* , while ^{2} in X_1^2, *etc.* is "*power* 2 " !



The Chi-Square *density* and *distribution* functions for $n = 1, 2, \cdots, 10$.

(In the Figure for F, the value of n increases from left to right.)

If $n = 1$ then

$$\chi_1^2 \quad \equiv \quad X_1^2 \ , \quad \text{where} \quad X \equiv X_1 \quad \text{is } \textit{standard normal} \ .$$

We can compute the *moment generating function* of χ_1^2 :

$$E[e^{t\chi_1^2}] \quad = \quad E[e^{tX^2}] \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{1}{2}x^2} \ dx$$

$$= \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2t)} \ dx$$

Let

$$1 - 2t \ = \ \frac{1}{\hat{\sigma}^2} \ , \quad \text{or equivalently} \ , \quad \hat{\sigma} \ \equiv \ \frac{1}{\sqrt{1-2t}} \ .$$

Then

$$E[e^{t\chi_1^2}] \quad = \quad \hat{\sigma} \cdot \frac{1}{\sqrt{2\pi}\ \hat{\sigma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2/\hat{\sigma}^2} \ dx \quad = \quad \hat{\sigma} \quad = \quad \frac{1}{\sqrt{1-2t}} \ .$$

(integral of a normal density function)

Thus we have found that :

The *moment generating function* of χ_1^2 is

$$\psi(t) \quad \equiv \quad E[e^{t\chi_1^2}] \quad = \quad \frac{1}{\sqrt{1-2t}} \ ,$$

with which we can compute

$$E[\chi_1^2] \qquad = \quad \psi'(0) \quad = \quad 1 \ , \qquad (\text{ Check ! })$$

$$E[(\chi_1^2)^2] \quad = \quad \psi''(0) \quad = \quad 3 \ , \qquad (\text{ Check ! })$$

$$Var(\chi_1^2) \quad = \quad E[(\chi_1^2)^2] \ - \ E[\chi_1^2]^2 \quad = \quad 2 \ .$$

We found that

$$E[\chi_1^2] \;=\; 1 \qquad , \qquad Var(\chi_1^2) \;=\; 2 \; .$$

In the *general case* where

$$\chi_n^2 \;\equiv\; X_1^2 \,+\, X_2^2 \,+\, \cdots \,+\, X_n^2 \;,$$

we have

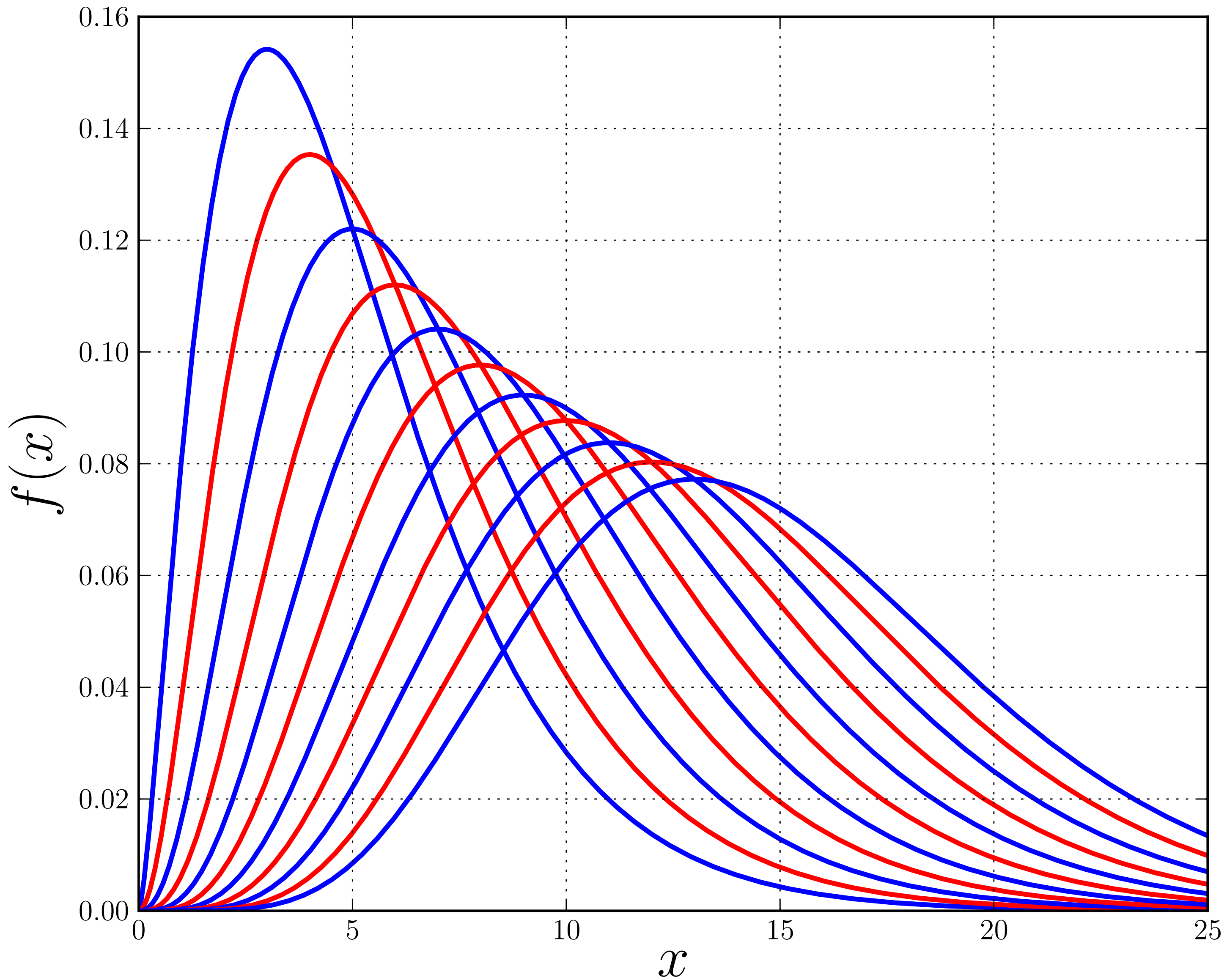
$$E[\chi_n^2] \;=\; E[X_1^2] \,+\, E[X_2^2] \,+\, \cdots \,+\, E[X_n^2] \;=\; n \;,$$

and since the X_i are assumed *independent* ,

$$Var[\chi_n^2] \;=\; Var[X_1^2] \,+\, Var[X_2^2] \,+\, \cdots \,+\, Var[X_n^2] \;=\; 2n \;,$$

and

$$\sigma(\chi_n^2) \;=\; \sqrt{2n} \; .$$



The Chi-Square *density* functions for $n = 5, 6, \cdots, 15$.

(For *large* n they look like *normal* density functions !)

The χ^2_n - Table

n	$\alpha = 0.975$	$\alpha = \mathbf{0.95}$	$\alpha = 0.05$	$\alpha = 0.025$
5	0.83	1.15	11.07	12.83
6	1.24	1.64	12.59	14.45
7	1.69	2.17	14.07	16.01
8	2.18	2.73	15.51	17.54
9	2.70	3.33	16.92	19.02
10	3.25	**3.94**	18.31	20.48
11	3.82	4.58	19.68	21.92
12	4.40	5.23	21.03	23.34
13	5.01	5.89	22.36	24.74
14	5.63	6.57	23.69	26.12
15	6.26	7.26	25.00	27.49

This Table shows $z_{\alpha,n}$ values such that $P(\chi^2_n \geq z_{\alpha,n}) = \alpha$.

(For example, $P(\chi^2_{\mathbf{10}} \geq \mathbf{3.94}) = \mathbf{95}\%$)

THE CENTRAL LIMIT THEOREM

The density function of the Chi-Square random variable

$$\chi_n^2 \quad \equiv \quad \tilde{X}_1 \; + \; \tilde{X}_2 \; + \; \cdots \; + \; \tilde{X}_n \; ,$$

where

$$\tilde{X}_i = X_i^2 \; , \qquad \text{and} \qquad X_i \; \text{ is standard normal, } \; i = 1, 2, \cdots, n \; ,$$

starts looking like a *normal density function* when n gets large.

- This remarkable fact holds much more generally !

- It is known as the *Central Limit Theorem* (CLT).

RECALL :

If X_1 , X_2 , $\cdots$, X_n are *independent, identically distributed* ,

each having

$$\text{mean } \mu \quad , \quad \text{variance } \sigma^2 \quad , \quad \text{standard deviation } \sigma \ ,$$

then

$$S \equiv X_1 + X_2 + \cdots + X_n \ ,$$

has

mean : $\mu_S \equiv E[S] = n\mu$ (Why ?)

variance : $Var(S) = n\sigma^2$ (Why ?)

Standard deviation : $\sigma_S = \sqrt{n}\,\sigma$

NOTE : σ_S gets *bigger* as n increases (and so does $|\mu_S|$).

THEOREM (The Central Limit Theorem) (CLT) :

Let X_1 , X_2 , $\cdots$, X_n be *identical, independent* random variables,

each having

$$\text{mean } \mu \quad , \quad \text{variance } \sigma^2 \quad , \quad \text{standard deviation } \sigma \ .$$

Then for *large* n the random variable

$$S \equiv X_1 + X_2 + \cdots + X_n \ ,$$

(which has mean $n\mu$, variance $n\,\sigma^2$, standard deviation $\sqrt{n}\,\sigma$)

is *approximately normal* .

NOTE : Thus $\dfrac{S - n\mu}{\sqrt{n}\,\sigma}$ is approximately *standard normal* .

EXAMPLE : Recall that

$$\chi_n^2 \quad \equiv \quad X_1^2 \ + \ X_2^2 \ + \ \cdots \ + \ X_n^2 \ ,$$

where each X_i is standard normal, and (using moments) we found

$$\chi_n^2 \text{ has } \textit{mean } n \quad \text{and} \quad \textit{standard deviation } \sqrt{2n} \ .$$

The Table below illustrates the accuracy of the approximation

$$P(\ \chi_n^2 \leq 0\ ) \quad \cong \quad \Phi(\ \frac{0-n}{\sqrt{2n}}\ ) \quad = \quad \Phi(-\sqrt{\frac{n}{2}}\ ) \ .$$

n	$-\sqrt{\frac{n}{2}}$	$\Phi(-\sqrt{\frac{n}{2}}\ )$
2	-1	0.1587
8	-2	0.0228
18	-3	0.0013

QUESTION : What is the exact value of $P(\chi_n^2 \leq 0)$? (!)

EXERCISE :

Use the approximation

$$P(\ \chi_n^2 \leq x\ ) \quad \cong \quad \Phi(\ \frac{x-n}{\sqrt{2n}}\ )\ ,$$

to compute approximate values of

- $P(\ \chi_{32}^2\ \leq\ 24\ )$

- $P(\ \chi_{32}^2\ \geq\ 40\ )$

- $P(\ \mid \chi_{32}^2 - 32 \mid\ \leq\ 8\ )$

RECALL :

If X_1 , X_2 , $\cdots$, X_n are *independent, identically distributed* ,

each having

$$\text{mean } \mu \ , \quad \text{variance } \sigma^2 \ , \quad \text{standard deviation } \sigma \ ,$$

then

$$\bar{X} \equiv \frac{1}{n} (X_1 + X_2 + \cdots + X_n) \ ,$$

has

mean : $\mu_{\bar{X}} = E[\bar{X}] = \mu$ (Why ?)

variance : $\sigma^2_{\bar{X}} = \frac{1}{n^2} n\sigma^2 = \sigma^2/n$ (Why ?)

Standard deviation : $\sigma_{\bar{X}} = \sigma/\sqrt{n}$

NOTE : $\sigma_{\bar{X}}$ gets *smaller* as n increases.

COROLLARY (to the Central Limit Theorem) :

If X_1 , X_2 , $\cdots$, X_n be *identical, independent* random variables,

each having

$$\text{mean} \quad \mu \quad , \quad \text{variance} \quad \sigma^2 \quad , \quad \text{standard deviation} \quad \sigma \ ,$$

then for *large* n the random variable

$$\bar{X} \quad \equiv \quad \frac{1}{n} (X_1 \ + \ X_2 \ + \ \cdots \ + \ X_n) \ ,$$

(which has mean μ , variance $\dfrac{\sigma^2}{n}$, standard deviation $\dfrac{\sigma}{\sqrt{n}}$)

is *approximately normal* .

NOTE : Thus $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ is approximately *standard normal* .

EXAMPLE : Suppose X_1 , X_2 , $\cdots$, X_n are

identical , *independent* , *uniform* random variables ,

each having density function

$$f(x) \;=\; \frac{1}{2}\,, \quad \text{for} \quad x \in [-1,1]\,, \qquad (\; 0 \;\; \text{otherwise} \;)\,,$$

with

mean $\mu = 0$, *standard deviation* $\sigma = \frac{1}{\sqrt{3}}$ (Check !)

Then for *large* n the random variable

$$\bar{X} \;\equiv\; \frac{1}{n}\,(X_1 \,+\, X_2 \,+\, \cdots \,+\, X_n)\,,$$

with

mean $\mu = 0$, *standard deviation* $\sigma = \frac{1}{\sqrt{3n}}$,

is *approximately normal* , so that

$$P(\bar{X} \le x) \;\cong\; \Phi\Big(\, \frac{x - 0}{1/\sqrt{3n}} \,\Big) \;\cong\; \Phi(\, \sqrt{3n}\; x \,)\,.$$

EXERCISE : In the preceding example

$$P(\bar{X} \leq x) \quad \cong \quad \Phi\left(\frac{x-0}{1/\sqrt{3n}} \right) \quad \equiv \quad \Phi(\sqrt{3n}\ x) \ .$$

- Fill in the Table to illustrate the accuracy of this approximation :

n	$P(\bar{X} \leq -1) \ \cong \ \Phi(-\sqrt{3n}\ )$
3	
12	

(What is the exact value of $P(\bar{X} \leq -1)$? !)

- For $n = 12$ find the approximate value of $P(\bar{X} \leq -0.1)$.

- For $n = 12$ find the approximate value of $P(\bar{X} \leq -0.5)$.

EXPERIMENT : (a lengthy one $\cdots$!)

We give a detailed *computational example* to illustrate :

- The concept of *density function* .

- The numerical *construction* of a density function

  and (most importantly)

- The *Central Limit Theorem* .

EXPERIMENT : (continued $\cdots$)

- Generate N *uniformly distributed* random numbers in $[0, 1]$.

- Many programming languages have a *function* for this.

- Call the random number values generated $\tilde{x}_i$, $i = 1, 2, \cdots, N$.

- Letting $x_i = 2\tilde{x}_i - 1$ gives *uniform random values in* $[-1, 1]$.

EXPERIMENT : (continued $\cdots$)

-0.737	0.511	-0.083	0.066	-0.562	-0.906	0.358	0.359
0.869	-0.233	0.039	0.662	-0.931	-0.893	0.059	0.342
-0.985	-0.233	-0.866	-0.165	0.374	0.178	0.861	0.692
0.054	-0.816	0.308	-0.168	0.402	0.821	0.524	-0.475
-0.905	0.472	-0.344	0.265	0.513	0.982	-0.269	-0.506
0.965	0.445	0.507	0.303	-0.855	0.263	0.769	-0.455
-0.127	0.533	-0.045	-0.524	-0.450	-0.281	-0.667	-0.027
0.795	0.818	-0.879	0.809	0.009	0.033	-0.362	0.973
-0.012	-0.468	-0.819	0.896	-0.853	0.001	-0.232	-0.446
0.828	0.059	-0.071	0.882	-0.900	0.523	0.540	0.656
-0.749	-0.968	0.377	0.736	0.259	0.472	0.451	0.999
0.777	-0.534	-0.387	-0.298	0.027	0.182	0.692	-0.176
0.683	-0.461	-0.169	0.075	-0.064	-0.426	-0.643	-0.693
0.143	0.605	-0.934	0.069	-0.003	0.911	0.497	0.109
0.781	0.250	0.684	-0.680	-0.574	0.429	-0.739	-0.818

120 values of a *uniform random variable* in $[-1, 1]$.

EXPERIMENT : (continued $\cdots$)

- Divide $[-1,1]$ into M subintervals of equal size $\Delta x = \frac{2}{M}$.

- Let I_k denote the kth interval, with midpoint x_k .

- Let m_k be the *frequency count* (# of random numbers in I_k) .

- Let $f(x_k) = \frac{m_k}{N \ \Delta x}$, (N is the total # of random numbers) .

- Then $\int_{-1}^{1} f(x)\, dx \cong \sum_{k=1}^{M} f(x_k)\, \Delta x = \sum_{k=1}^{M} \frac{m_k}{N \ \Delta x} \Delta x = 1$,

  and $f(x_k)$ approximates the value of the *density function* .

- The corresponding *distribution function* is

$$F(x_\ell) = \int_{-1}^{x_\ell} f(x)\, dx \cong \sum_{k=1}^{\ell} f(x_k)\, \Delta x .$$

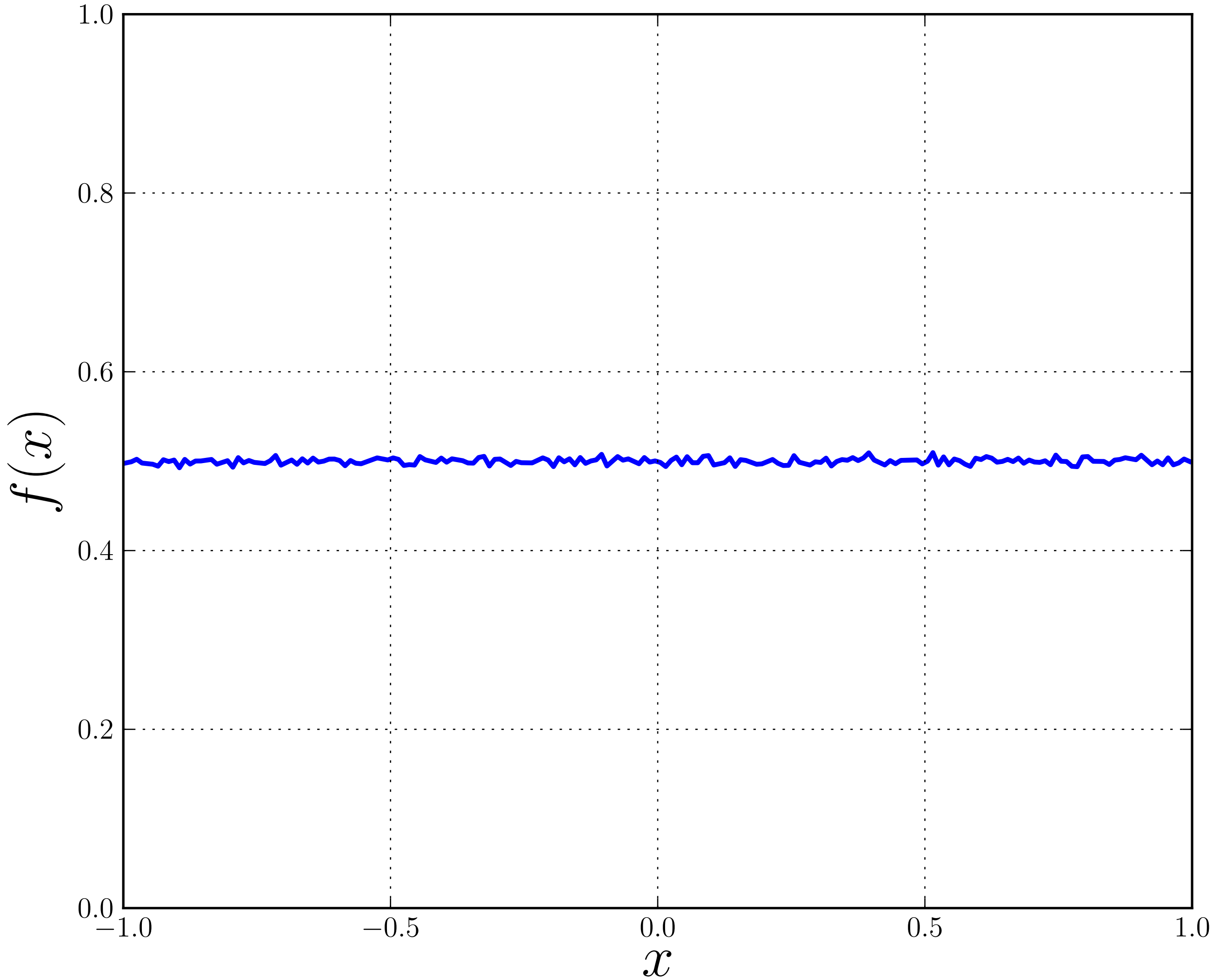
EXPERIMENT : (continued $\cdots$)

Interval	Frequency	Sum	$f(x)$	$F(x)$
1	50013	50013	0.500	0.067
2	50033	100046	0.500	0.133
3	50104	150150	0.501	0.200
4	49894	200044	0.499	0.267
5	50242	250286	0.502	0.334
6	49483	299769	0.495	0.400
7	50016	349785	0.500	0.466
8	50241	400026	0.502	0.533
9	50261	450287	0.503	0.600
10	49818	500105	0.498	0.667
11	49814	549919	0.498	0.733
12	50224	600143	0.502	0.800
13	49971	650114	0.500	0.867
14	49873	699987	0.499	0.933
15	50013	750000	0.500	1.000

Frequency Table, showing the count per interval .

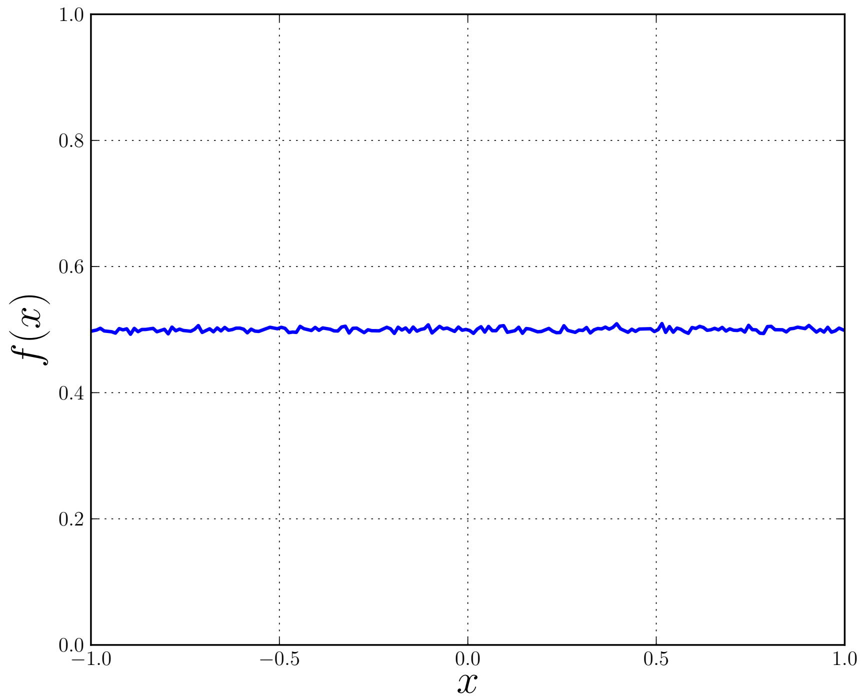
($N = 750,000$ random numbers, $M = 15$ intervals)

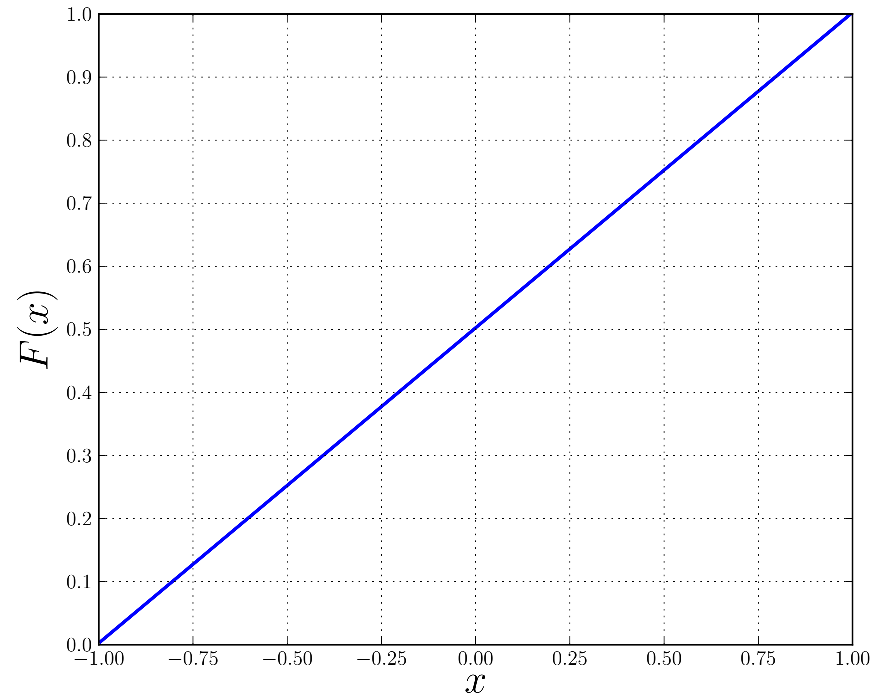
EXPERIMENT : (continued $\cdots$)



The approximate *density function* , $f(x_k) = \frac{m_k}{N \ \Delta x}$ for $N = 5,000,000$ random numbers, and $M = 200$ intervals.

EXPERIMENT : (continued $\cdots$)





Approximate *density function* $f(x)$ and *distribution function* $F(x)$, for the case $N = 5,000,000$ random numbers, and $M = 200$ intervals.

NOTE : $F(x)$ appears *smoother* than $f(x)$. (Why ?)

EXPERIMENT : (continued $\cdots$)

Next $\cdots$ (still for the uniform random variable in $[-1, 1]$) :

- Generate n *random numbers* (n relatively *small*) .

- Take the *average* of the n random numbers.

- Do the above N times, where (as before) N is *very large* .

- Thus we deal with a random variable

$$\bar{X} \equiv \frac{1}{n}(X_1 + X_2 + \cdots + X_n) .$$

EXPERIMENT : (continued $\cdots$)

-0.047	0.126	-0.037	0.148	-0.130	-0.004	-0.174	0.191
0.198	0.073	-0.025	-0.070	-0.018	-0.031	0.063	-0.064
-0.197	-0.026	-0.062	-0.004	-0.083	-0.031	-0.102	-0.033
-0.164	0.265	-0.274	0.188	-0.067	0.049	-0.090	0.002
0.118	0.088	-0.071	0.067	-0.134	-0.100	0.132	0.242
-0.005	-0.011	-0.018	-0.048	-0.153	0.016	0.086	-0.179
-0.011	-0.058	0.198	-0.002	0.138	-0.044	-0.094	0.078
-0.011	-0.093	0.117	-0.156	-0.246	0.071	0.166	0.142
0.103	-0.045	-0.131	-0.100	0.072	0.034	0.176	0.108
0.108	0.141	-0.009	0.140	0.025	-0.149	0.121	-0.120
0.012	0.002	-0.015	0.106	0.030	-0.096	-0.024	-0.111
-0.147	0.004	0.084	0.047	-0.048	0.018	-0.183	0.069
-0.236	-0.217	0.061	0.092	-0.003	0.005	-0.054	0.025
-0.110	-0.094	-0.115	0.052	0.135	-0.076	-0.018	-0.121
-0.030	-0.146	-0.155	0.089	-0.177	0.027	-0.025	0.020

Values of $\bar{X}$ for the case $N = 120$ and $n = 25$.

EXPERIMENT : (continued $\cdots$)

For sample size n, $(n = 1, 2, 5, 10, 25)$, and $M = 200$ intervals :

- Generate N values of $\bar{X}$, where *N is very large* .

- Let m_k be *the number of values of $\bar{X}$ in I_k* .

- As before, let $f_n(x_k) = \frac{m_k}{N\ \Delta x}$.

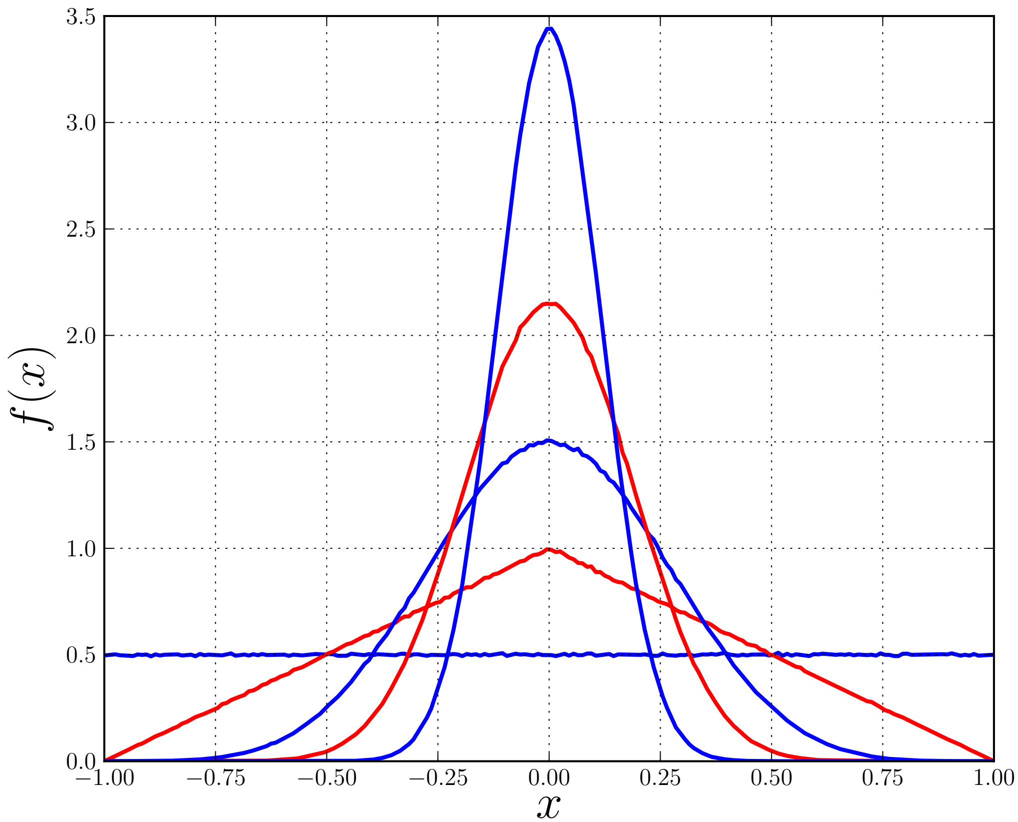
- Now $f_n(x_k)$ approximates the *density function of $\bar{X}$* .

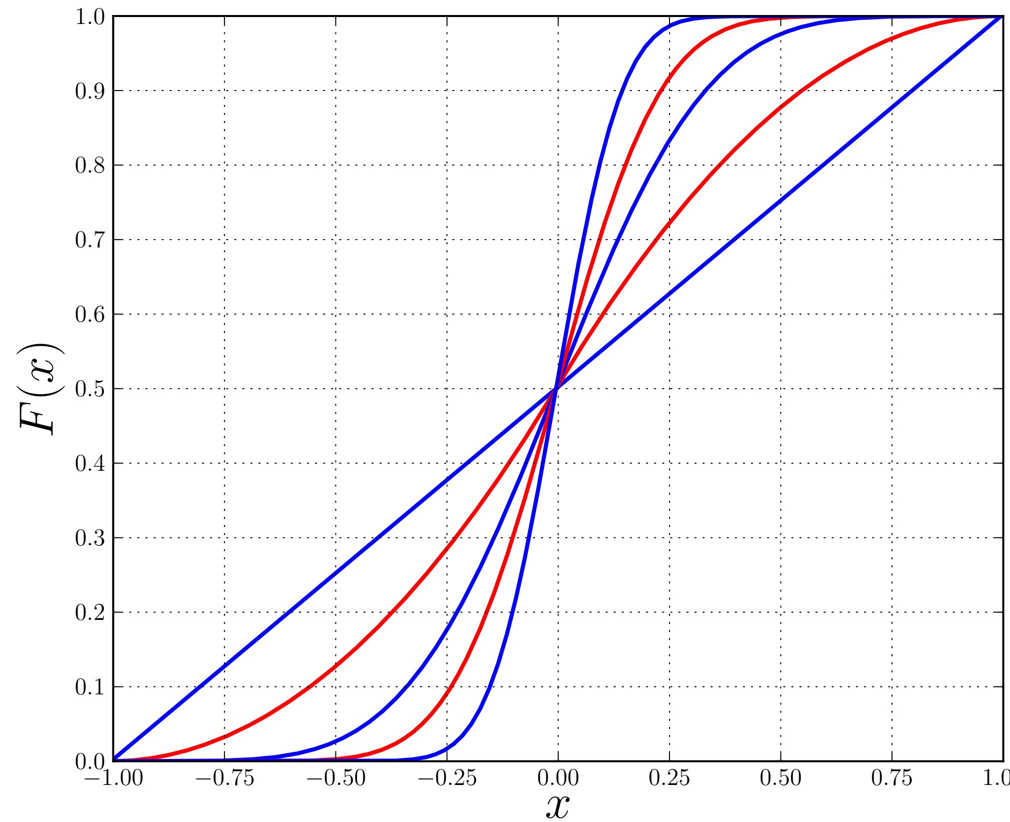
EXPERIMENT : (continued $\cdots$)

Interval	Frequency	Sum	$f(x)$	$F(x)$
1	0	0	0.00000	0.00000
2	0	0	0.00000	0.00000
3	0	0	0.00000	0.00000
4	11	11	0.00011	0.00001
5	1283	1294	0.01283	0.00173
6	29982	31276	0.29982	0.04170
7	181209	212485	1.81209	0.28331
8	325314	537799	3.25314	0.71707
9	181273	719072	1.81273	0.95876
10	29620	748692	0.29620	0.99826
11	1294	749986	0.01294	0.99998
12	14	750000	0.00014	1.00000
13	0	750000	0.00000	1.00000
14	0	750000	0.00000	1.00000
15	0	750000	0.00000	1.00000

Frequency Table for $\bar{X}$, showing the count per interval .
($N = 750,000$ values of $\bar{X}$, $M = 15$ intervals, sample size $n = 25$)

EXPERIMENT : (continued ···)





The approximate *density functions* $f_n(x)$, $n = 1,\ 2,\ 5,\ 10,\ 25$,

and the corresponding *distribution functions* $F_n(x)$.

($N = 5,000,000$ values of $\bar{X}$, $M = 200$ intervals)

EXPERIMENT : (continued $\cdots$)

Recall that for *uniform random variables* X_i on $[-1,1]$

$$\bar{X} \quad \equiv \quad \frac{1}{n} \left(X_1 \ + \ X_2 \ + \ \cdots \ + \ X_n \right) \ ,$$

is *approximately normal* , with

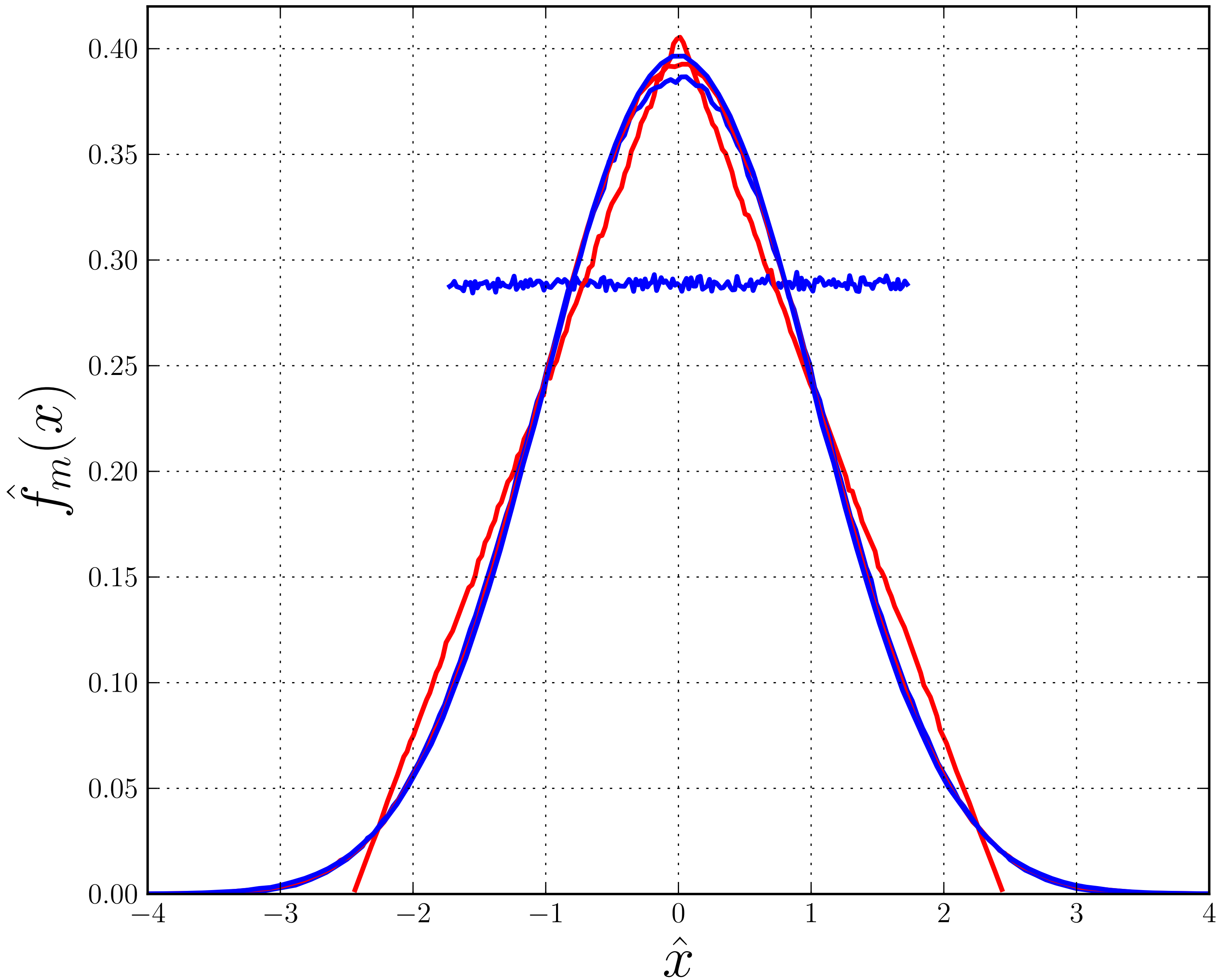
$$\textit{mean} \quad \mu = 0 \qquad , \qquad \textit{standard deviation} \quad \sigma = \frac{1}{\sqrt{3n}} \ .$$

Thus for each n we can *normalize* x and $f_n(x)$:

$$\hat{x} \quad = \quad \frac{x - \mu}{\sigma} \quad = \quad \frac{x - 0}{\frac{1}{\sqrt{3n}}} \quad = \quad \sqrt{3n}\ x \qquad , \qquad \hat{f}_n(\hat{x}) \quad = \quad \frac{f_n(x)}{\sqrt{3n}} \ .$$

The next Figure shows :

- The normalized $\hat{f}_n(\hat{x})$ approach a limit as n get large.
- This limit is the *standard normal* density function.
- Thus our computations agree with the Central Limit Theorem !



The normalized density functions $\hat{f}_n(x)$, for $n = 1, 2, 5, 10, 25$.

($N = 5,000,000$ values of $\bar{X}$, $M = 200$ intervals)

EXERCISE : Suppose

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_{12} \ , \qquad (n = 12) \ ,$$

are identical, independent, *uniform* random variables on $[0,1]$.

We already know that each X_i has

$$\text{mean} \ \ \mu = \tfrac{1}{2} \qquad , \qquad \text{standard deviation} \ \ \tfrac{1}{2\sqrt{3}} \ .$$

Let

$$\bar{X} \quad \equiv \quad \frac{1}{12} \left(X_1 \ + \ X_2 \ + \ \cdots \ + \ X_{12} \right) \ .$$

Use the CLT to *compute approximate values* of

- $P(\bar{X} \ \leq \ \frac{1}{3})$
- $P(\bar{X} \ \geq \ \frac{2}{3})$
- $P(| \ \bar{X} - \frac{1}{2} \ | \ \ \leq \ \frac{1}{3})$

EXERCISE : Suppose

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_9 \ , \qquad (n = 9) \ ,$$

are identical, independent, *exponential* random variables, with

$$f(x) \quad = \quad \lambda e^{-\lambda x} \ , \quad \text{where} \quad \lambda = 1 \ .$$

We already know that each X_i has

$$\text{mean} \quad \mu = \tfrac{1}{\lambda} = 1 \ , \qquad \text{and} \quad \text{standard deviation} \quad \tfrac{1}{\lambda} = 1 \ .$$

Let

$$\bar{X} \quad \equiv \quad \frac{1}{9} \, (X_1 \ + \ X_2 \ + \ \cdots \ + \ X_9) \ .$$

Use the CLT to *compute approximate values* of

- $P(\bar{X} \ \leq \ 0.4)$
- $P(\bar{X} \ \geq \ 1.6)$
- $P(|\ \bar{X} - 1 \ | \ \leq \ 0.6)$

EXERCISE : Suppose

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ ,$$

are identical, independent, *normal* random variables, with

$$\text{mean} \ \ \mu = 7 \qquad , \qquad \text{standard deviation} \ \ 4 \ .$$

Let

$$\bar{X} \quad \equiv \quad \frac{1}{n} \left(X_1 \ + \ X_2 \ + \ \cdots \ + \ X_n \right) \ .$$

Use the CLT to determine at least how big n must be so that

- $P(| \bar{X} - \mu | \ \le \ 1) \ \ge \ 90 \ \% \ .$

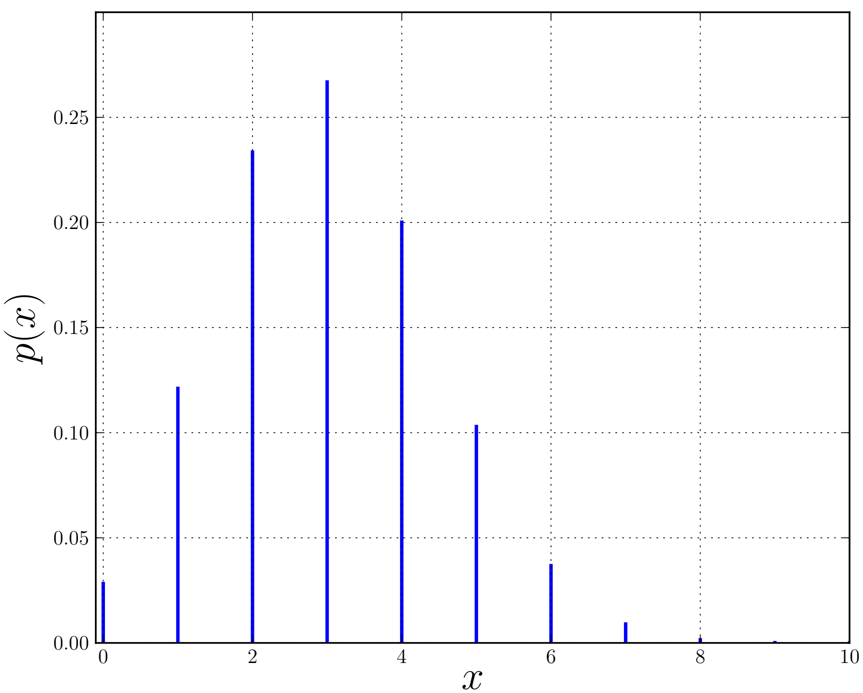
EXAMPLE : The CLT also applies to *discrete random variables* .

The *Binomial random variable* , with

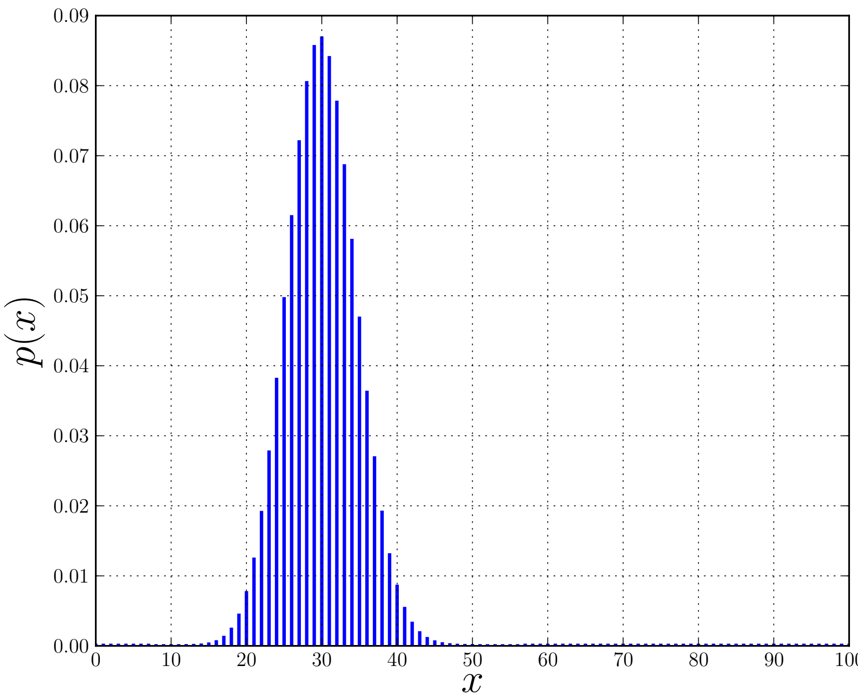
$$P(X = k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \ , \qquad (0 \leq k \leq n) \ ,$$

is already a *sum* (namely, of *Bernoulli* random variables).

Thus its *binomial probability mass function* already "*looks normal*" :



Binomial : $n = 10$, $p = 0.3$



Binomial : $n = 100$, $p = 0.3$

EXAMPLE : (continued $\cdots$)

We already know that if X is *binomial* then

$$\mu(X) \; = \; np \qquad \text{and} \qquad \sigma(X) \; = \; \sqrt{np(1-p)} \; .$$

Thus, for $n = 100$, $p = 0.3$, we have

$$\mu(X) \; = \; 30 \qquad \text{and} \qquad \sigma(X) \; = \; \sqrt{21} \; \cong \; 4.58 \; .$$

Using the CLT we can *approximate*

$$P(X \leq 26) \; \cong \; \Phi(\frac{26-30}{4.58}) \; = \; \Phi(-0.87) \; \cong \; \mathbf{19.2}\,\% \; .$$

The *exact binomial value* is

$$P(X \leq 26) \; = \; \sum_{k=0}^{26} \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \; = \; \mathbf{22.4}\,\% \; ,$$

QUESTION : What do you say ?

EXAMPLE : (continued $\cdots$)

We found the *exact* binomial value

$$P(X \leq 26) \quad = \quad \mathbf{22}.\mathbf{4} \ \% \ ,$$

and the CLT *approximation*

$$P(X \leq 26) \quad \cong \quad \Phi(\frac{26-30}{4.58}) \quad = \quad \Phi(-0.87) \quad \cong \quad 19.2 \ \% \ .$$

It is *better* to

"*spread*" $P(X = 26)$ *over the interval* $[25.5 \ , \ 26.5]$. (Why ?)

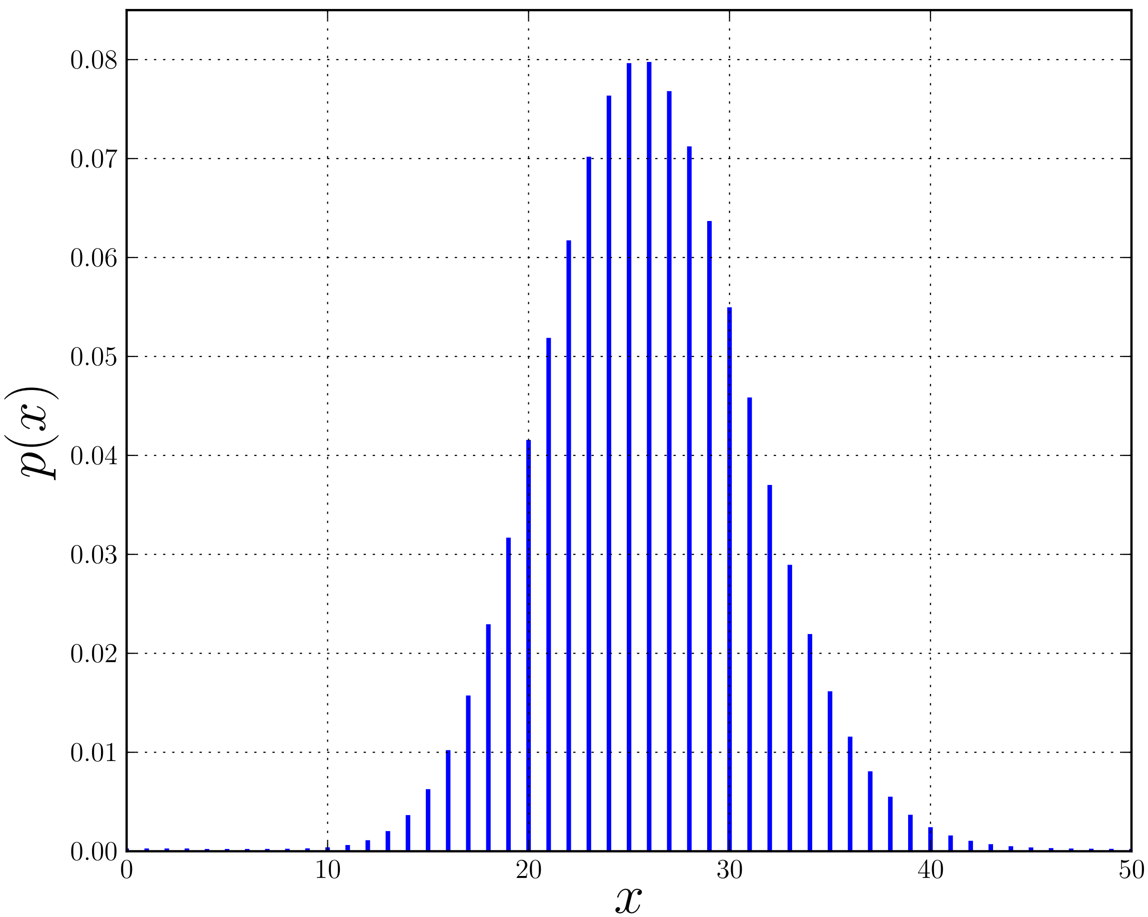
Thus it is *better* to *adjust* the approximation to $P(X \leq 26)$ by

$$P(X \leq 26) \quad \cong \quad \Phi(\frac{26.5-30}{4.58}) \quad = \quad \Phi(-0.764) \quad \cong \quad \mathbf{22}.\mathbf{2} \ \% \ .$$

QUESTION : What do you say now ?

EXERCISE :

Consider the *Binomial* distribution with $n = 676$ and $p = \frac{1}{26}$:



The Binomial $(n = 676,\ p = \frac{1}{26})$, shown in $[0, 50]$.

EXERCISE : (continued $\cdots$) (Binomial : $n = 676$, $p = \frac{1}{26}$)

- Write down the *Binomial formula* for $P(X = 24)$.
- Evaluate $P(X = 24)$ using the Binomial *recurrence formula* .
- Compute $E[X] = np$ and $\sigma(X) = \sqrt{np(1-p)}$.

The *Poisson* probability mass function

$$P(X = k) \; = \; e^{-\lambda} \, \frac{\lambda^k}{k!} \, , \qquad (\text{with } \; \lambda = np \;) \, ,$$

approximates the Binomial when p is small and n large.

- Evaluate $P(X = 24)$ using the Poisson *recurrence formula* .

- Compute the *standard normal* approximation to $P(X = 24)$.

ANSWERS : 7.61 % , 7.50 % , 7.36 % .

EXPERIMENT :

Compare the *accuracy* of the Poisson and the adjusted Normal *approximations* to the Binomial, for different values of n .

k	n	Binomial	Poisson	Normal
2	4	0.6875	0.6767	0.6915
4	8	0.6367	0.6288	0.6382
8	16	0.5982	0.5925	0.5987
16	32	0.5700	0.5660	0.5702
32	64	0.5497	0.5468	0.5497
64	128	0.5352	0.5332	0.5352

$P(X \leq k)$, where $k = \lfloor np \rfloor$, with $p = 0.5$.

- Any conclusions ?

EXPERIMENT : (continued $\cdots$)

Compare the *accuracy* of the Poisson and the adjusted Normal *approximations* to the Binomial, for different values of n .

k	n	Binomial	Poisson	Normal
0	4	0.6561	0.6703	0.5662
0	8	0.4305	0.4493	0.3618
1	16	0.5147	0.5249	0.4668
3	32	0.6003	0.6025	0.5702
6	64	0.5390	0.5423	0.5166
12	128	0.4805	0.4853	0.4648
25	256	0.5028	0.5053	0.4917
51	512	0.5254	0.5260	0.5176

$P(X \leq k)$, where $k = \lfloor np \rfloor$, with $p = 0.1$.

- Any conclusions ?

EXPERIMENT : (continued $\cdots$)

Compare the *accuracy* of the Poisson and the adjusted Normal *approximations* to the Binomial, for different values of n .

k	n	Binomial	Poisson	Normal
0	4	0.9606	0.9608	0.9896
0	8	0.9227	0.9231	0.9322
0	16	0.8515	0.8521	0.8035
0	32	0.7250	0.7261	0.6254
0	64	0.5256	0.5273	0.4302
1	128	0.6334	0.6339	0.5775
2	256	0.5278	0.5285	0.4850
5	512	0.5948	0.5949	0.5670
10	1024	0.5529	0.5530	0.5325
20	2048	0.5163	0.5165	0.5018
40	4096	0.4814	0.4817	0.4712

$P(X \leq k)$, where $k = \lfloor np \rfloor$, with $p = 0.01$.

- Any conclusions ?

SAMPLE STATISTICS

Sampling can consist of

- *Gathering random data* from a large *population*, for example,
  - measuring the height of randomly selected adults
  - measuring the starting salary of random CS graduates

- Recording the *results of experiments* , for example,
  - measuring the breaking strength of randomly selected bolts
  - measuring the lifetime of randomly selected light bulbs

- We shall generally assume the population is *infinite* (or *large*) .
- We shall also generally assume the observations are *independent* .
- The outcome of any experiment does not affect other experiments.

DEFINITIONS :

- A *random sample* from a population consists of

  independent , *identically distributed* random variables,

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ .$$

- The values of the X_i are called the *outcomes* of the experiment.

- A *statistic* is a *function* of $X_1, \ X_2, \ \cdots \ , X_n$.

- Thus a *statistic* itself is a *random variable* .

EXAMPLES :

The most *important statistics* are

- The *sample mean*

$$\bar{X} \equiv \frac{1}{n}(X_1 + X_2 + \cdots + X_n) .$$

- The *sample variance*

$$S^2 \equiv \frac{1}{n}\sum_{k=1}^{n}(X_k - \bar{X})^2 .$$

(to be discussed in detail $\cdots$)

- The *sample standard deviation* $S = \sqrt{S^2}$.

For a random sample

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ ,$$

one can think of many other *statistics* such as :

- The *order statistic* in which the observation are *ordered in size* .

- The *sample median* , which is
  - the *midvalue of the order statistic* (if n is odd),
  - the *average* of the *two middle values* (if n is even).

- The *sample range* : the difference between the largest and the smallest observation.

EXAMPLE : For the 8 observations

$$-0.737\ ,\ 0.511\ ,-0.083\ ,\ 0.066\ ,-0.562\ ,-0.906\ ,\ 0.358\ ,\ 0.359\ ,$$

from the first row of the Table given earlier, we have

Sample mean :

$$\bar{X} \;=\; \frac{1}{8}\,(\;-0.737 \;+\; 0.511 \;-\; 0.083 \;+\; 0.066$$
$$-\; 0.562 \;-\; 0.906 \;+\; 0.358 \;+\; 0.359\;) \;=\; -0.124\ .$$

Sample variance :

$$\frac{1}{8}\{\;(-0.737-\bar{X})^2 \;+\; (0.511-\bar{X})^2 \;+\; (-0.083-\bar{X})^2$$
$$+\;(0.066-\bar{X})^2 \;+\; (-0.562-\bar{X})^2 \;+\; (-0.906-\bar{X})^2$$
$$+\;(0.358-\bar{X})^2 \;+\; (0.359-\bar{X})^2\;\} \;=\; 0.26\ .$$

Sample standard deviation : $\sqrt{0.26} \;=\; 0.51$.

EXAMPLE : (continued $\cdots$)

For the 8 observations

$$-0.737\ ,\ 0.511\ ,-0.083\ ,\ 0.066\ ,-0.562\ ,-0.906\ ,\ 0.358\ ,\ 0.359\ ,$$

we also have

The *order statistic* :

$$-0.906\ ,-0.737\ ,-0.562\ ,-0.083\ ,\ 0.066\ ,\ 0.358\ ,\ 0.359\ ,\ 0.511\ .$$

The *sample median* : $(-0.083 + 0.066)/2 \ = \ -0.0085$.

The *sample range* : $0.511 - (-0.906) \ = \ 1.417$.

The Sample Mean

Suppose the *population mean* and *standard deviation* are μ and σ .

As before, the *sample mean*

$$\bar{X} \equiv \frac{1}{n}(X_1 + X_2 + \cdots + X_n) ,$$

is also a *random variable* , with *expected value*

$$\mu_{\bar{X}} \equiv E[\bar{X}] = E[\frac{1}{n}(X_1 + X_2 + \cdots + X_n)] = \mu ,$$

and *variance*

$$\sigma_{\bar{X}}^2 \equiv Var(\bar{X}) = \frac{\sigma^2}{n} ,$$

$$\textit{Standard deviation of } \bar{X} : \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} .$$

NOTE : The *sample mean* approximates the *population mean* μ .

How well does the *sample mean* approximate the *population mean* ?

From the Corollary to the CLT we know

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}},$$

is approximately *standard normal* when n is large.

Thus, for given n and z , $(z > 0)$, we can, for example, estimate

$$P(\ | \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} | \ \leq \ z \ ) \ \cong \ 1 \ - \ 2 \ \Phi(-z) \ .$$

(A problem is that we often don't know the value of σ $\cdots$)

It follows that

$$P\Big(\ \Big| \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \Big| \ \leq \ z \ \Big) \ = \ P\Big(\ | \, \bar{X} - \mu \, | \ \leq \ \frac{\sigma z}{\sqrt{n}} \ \Big)$$

$$= \ P\Big(\ \mu \in [\ \bar{X} - \frac{\sigma z}{\sqrt{n}} \ , \ \bar{X} + \frac{\sigma z}{\sqrt{n}} \ ] \ \Big)$$

$$\cong \ 1 \ - \ 2 \ \Phi(-z) \ ,$$

which gives us a *confidence interval estimate* of μ .

We found : $$P(\ \mu \in [\bar{X} - \frac{\sigma z}{\sqrt{n}}\ ,\ \bar{X} + \frac{\sigma z}{\sqrt{n}}]\ )\ \cong\ 1 - 2\ \Phi(-z)\ .$$

EXAMPLE : We take samples from a given population :

- The population *mean* μ is *unknown* .
- The population standard deviation is $\sigma = 3$
- The sample size is $n = 25$.
- The sample mean is $\bar{X} = 4.5$.

Taking **z=2** , we have

$$P(\ \mu \in [\ 4.5 - \frac{3 \cdot \mathbf{2}}{\sqrt{25}}\ ,\ 4.5 + \frac{3 \cdot \mathbf{2}}{\sqrt{25}}\ ]\ )\ =\ P(\ \mu \in [\ 3.3\ ,\ 5.7\ ]\ )$$

$$\cong\ 1 - 2\ \Phi(-\mathbf{2})\ \cong\ 95\ \%\ .$$

We call $[\ 3.3\ ,\ 5.7\ ]$ the $95\ \%$ *confidence interval estimate* of μ .

EXERCISE :

As in the preceding example, μ is unknown, $\sigma = 3$, $\bar{X} = 4.5$.

Use the formula

$$P(\ \mu \in [\bar{X} - \frac{\sigma z}{\sqrt{n}}\ ,\ \bar{X} + \frac{\sigma z}{\sqrt{n}}]\ )\ \cong\ 1 - 2\ \Phi(-z)\ ,$$

to determine

- The 50 % *confidence interval estimate* of μ when $n = 25$.
- The 50 % *confidence interval estimate* of μ when $n = 100$.
- The 95 % *confidence interval estimate* of μ when $n = 100$.

NOTE : In the *Standard Normal Table* , check that

- The 50 % confidence interval corresponds to $z\ =\ 0.68\ \cong\ 0.7$.
- The 95 % confidence interval corresponds to $z\ =\ 1.96\ \cong\ 2.0$.

The Sample Variance We defined the *sample variance* as

$$S^2 \equiv \frac{1}{n}\sum_{k=1}^{n}(X_k-\bar{X})^2 = \sum_{k=1}^{n}\left[\,(X_k-\bar{X})^2\cdot\frac{1}{n}\,\right].$$

Earlier, for *discrete* random variables X, we defined the *variance* as

$$\sigma^2 \equiv E[\,(X-\mu)^2] \equiv \sum_{k}\left[\,(X_k-\mu)^2\cdot p(X_k)\,\right].$$

- These two formulas look *deceptively* similar !
- In fact, they are quite different !
- The 1st sum for S^2 is *only* over the *sampled* X-values.
- The 2nd sum for σ^2 is over *all* X-values.
- The 1st sum for S^2 has *constant weights* .
- The 2nd sum for σ^2 uses the *probabilities as weights* .

We have just argued that the *sample variance*

$$S^2 \quad \equiv \quad \frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2 \ ,$$

and the *population variance* (for *discrete* random variables)

$$\sigma^2 \quad \equiv \quad E[\ (X - \mu)^2] \quad \equiv \quad \sum_k \ [\ (X_k - \mu)^2 \ \cdot \ p(X_k)\ ]\ ,$$

are quite different.

Nevertheless, we will show that *for large* n *their values are close* !

Thus for large n we have the approximation

$$S^2 \quad \cong \quad \sigma^2\ .$$

FACT 1 : We (obviously) have that

$$\bar{X} \ = \ \frac{1}{n} \sum_{k=1}^{n} X_k \qquad \text{implies} \qquad \sum_{k=1}^{n} X_k \ = \ n\bar{X} \ .$$

FACT 2 : From

$$\sigma^2 \ \equiv \ Var(X) \ \equiv \ E[(X-\mu)^2] \ = \ E[X^2] \ - \ \mu^2 \ ,$$

we (obviously) have

$$E[X^2] \ = \ \sigma^2 \ + \ \mu^2 \ .$$

FACT 3 : Recall that for *independent, identically distributed* X_k ,

where each X_k has *mean* μ and *variance* σ^2 , we have

$$\mu_{\bar{X}} \ \equiv \ E[\bar{X}] \ = \ \mu \qquad , \qquad \sigma^2_{\bar{X}} \ \equiv \ E[(\bar{X}-\mu)^2] \ = \ \frac{\sigma^2}{n} \ .$$

FACT 4 : (Useful for computing S^2 efficiently) :

$$S^2 \quad \equiv \quad \frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2 \quad = \quad \frac{1}{n} \, [\, \sum_{k=1}^{n} X_k^2 \,] \quad - \quad \bar{X}^2 \, .$$

PROOF :

$$S^2 \quad = \quad \frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2$$

$$= \quad \frac{1}{n} \sum_{k=1}^{n} (X_k^2 - 2X_k\bar{X} + \bar{X}^2)$$

$$= \quad \frac{1}{n} \, [\sum_{k=1}^{n} X_k^2 \; - \; 2\bar{X} \sum_{k=1}^{n} X_k \; + \; n\bar{X}^2] \qquad \text{(now use Fact 1)}$$

$$= \quad \frac{1}{n} \, [\sum_{k=1}^{n} X_k^2 \; - \; 2n\bar{X}^2 \; + \; n\bar{X}^2] \quad = \quad \frac{1}{n} \, [\sum_{k=1}^{n} X_k^2] \quad - \quad \bar{X}^2$$ **QED** !

THEOREM : The sample variance

$$S^2 \equiv \frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2$$

has *expected value*

$$E[S^2] = (1 - \frac{1}{n}) \cdot \sigma^2 .$$

PROOF :

$$E[S^2] = E[\frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2]$$

$$= E\Big[\frac{1}{n} \sum_{k=1}^{n} [X_k^2] - \bar{X}^2 \Big] \qquad (\text{ using Fact 4 })$$

$$= \frac{1}{n} \sum_{k=1}^{n} E[X_k^2] - E[\bar{X}^2]$$

$$= \sigma^2 + \mu^2 - (\sigma_{\bar{X}}^2 + \mu_{\bar{X}}^2) \qquad (\text{ using Fact 2 } \quad n+1 \text{ times ! })$$

$$= \sigma^2 + \mu^2 - (\frac{\sigma^2}{n} + \mu^2) = (1 - \frac{1}{n})\, \sigma^2 . \quad (\text{ Fact 3 }) \qquad \textbf{QED !}$$

REMARK : Thus $\lim_{n\to\infty} E[S^2] = \sigma^2$.

Most authors instead define the *sample variance* as

$$\hat{S}^2 \equiv \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X})^2 .$$

In this case the Theorem becomes :

THEOREM : The sample variance

$$\hat{S}^2 \equiv \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X})^2$$

has *expected value*

$$E[\hat{S}^2] = \sigma^2 .$$

EXERCISE : Check this !

EXAMPLE : The *random sample* of 120 values of a *uniform random variable* on $[-1, 1]$ in an earlier Table has

$$\bar{X} = \frac{1}{120} \sum_{k=1}^{120} X_k = 0.030 ,$$

$$S^2 = \frac{1}{120} \sum_{k=1}^{120} (X_k - \bar{X})^2 = 0.335 ,$$

$$S = \sqrt{S^2} = 0.579 ,$$

while

$$\mu = 0 ,$$

$$\sigma^2 = \int_{-1}^{1} (x - \mu)^2 \frac{1}{2} dx = \frac{1}{3} ,$$

$$\sigma = \sqrt{\sigma^2} = \frac{1}{\sqrt{3}} = 0.577 .$$

- What do you say ?

EXAMPLE :

- Generate 50 *uniform random numbers* in $[-1, 1]$.

- Compute their average.

- Do the above 500 times.

- Call the results $\bar{X}_k$, $k = 1, 2, \cdots, 500$.

- Thus each $\bar{X}_k$ is the *average* of 50 random numbers.

- Compute the *sample statistics* $\bar{X}$ and S of these 500 values.

- Can you *predict* the values of $\bar{X}$ and S ?

EXAMPLE : (continued $\cdots$)

Results :

$$\bar{X} = \frac{1}{500}\sum_{k=1}^{500} \bar{X}_k = -0.00136 \ ,$$

$$S^2 = \frac{1}{500}\sum_{k=1}^{500} (\bar{X}_k - \bar{X})^2 = 0.00664 \ ,$$

$$S = \sqrt{S^2} = 0.08152 \ .$$

EXERCISE :

- What is the value of $E[\bar{X}]$?
- Compare $\bar{X}$ to $E[\bar{X}]$.
- What is the value of $Var(\bar{X})$?
- Compare S^2 to $Var(\bar{X})$.

Estimating the variance of a normal distribution

We have shown that

$$S^2 \equiv \frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2 \cong \sigma^2 .$$

How good is this approximation for *normal random variables* X_k ?

To answer this we need :

FACT 5 :

$$\sum_{k=1}^{n} (X_k - \mu)^2 - \sum_{k=1}^{n} (X_k - \bar{X})^2 = n(\bar{X} - \mu)^2 .$$

PROOF :

$$\begin{aligned}
\text{LHS} &= \textstyle\sum_{k=1}^{n} \{ X_k^2 - 2X_k\mu + \mu^2 - X_k^2 + 2X_k\bar{X} - \bar{X}^2 \} \\
&= -2n\bar{X}\mu + n\mu^2 + 2n\bar{X}^2 - n\bar{X}^2 \\
&= n\bar{X}^2 - 2n\bar{X}\mu + n\mu^2 = \text{RHS} .
\end{aligned}$$

QED !

Rewrite Fact 5

$$\sum_{k=1}^{n}(X_k-\mu)^2 \quad - \quad \sum_{k=1}^{n}(X_k-\bar{X})^2 \quad = \quad n(\bar{X}-\mu)^2 \ ,$$

as

$$\sum_{k=1}^{n}\Big(\frac{X_k-\mu}{\sigma}\Big)^2 \quad - \quad \frac{n}{\sigma^2}\ \frac{1}{n}\ \sum_{k=1}^{n}(X_k-\bar{X})^2 \ = \ \Big(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\Big)^2 \ ,$$

and then as

$$\sum_{k=1}^{n} Z_k^2 \quad - \quad \frac{n}{\sigma^2}\ S^2 \quad = \quad Z^2 \ ,$$

where

$$S^2 \quad \text{is the sample variance} \ ,$$

and

Z and Z_k are *standard normal* because the X_k are *normal* .

Finally, we can write the above as

$$\frac{n}{\sigma^2}\ S^2 \quad = \quad \chi_n^2 \ - \ \chi_1^2 \ . \qquad (\ \text{Why ?}\ )$$

We have found that

$$\frac{n}{\sigma^2}\, S^2 \;=\; \chi_n^2 \;-\; \chi_1^2 \;.$$

THEOREM : For samples from a normal distribution :

$$\frac{n}{\sigma^2}\, S^2 \quad \text{has the} \quad \chi_{n-1}^2 \quad \text{distribution !}$$

PROOF : Omitted (and not as obvious as it might appear !) .

REMARK : If we use the alternate definition

$$\hat{S}^2 \;\equiv\; \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X})^2 \;,$$

then the Theorem becomes

$$\frac{n-1}{\sigma^2}\, \hat{S}^2 \quad \text{has the} \quad \chi_{n-1}^2 \quad \text{distribution .}$$

For normal random variables : $\frac{n-1}{\sigma^2}\,\hat{S}^2$ has the χ^2_{n-1} distribution

EXAMPLE : For a large shipment of light bulbs we know that :

- The lifetime of the bulbs has a *normal distribution* .
- The *standard deviation* is claimed to be $\sigma = \mathbf{100}$ hours.

  (The mean lifetime μ is not given.)

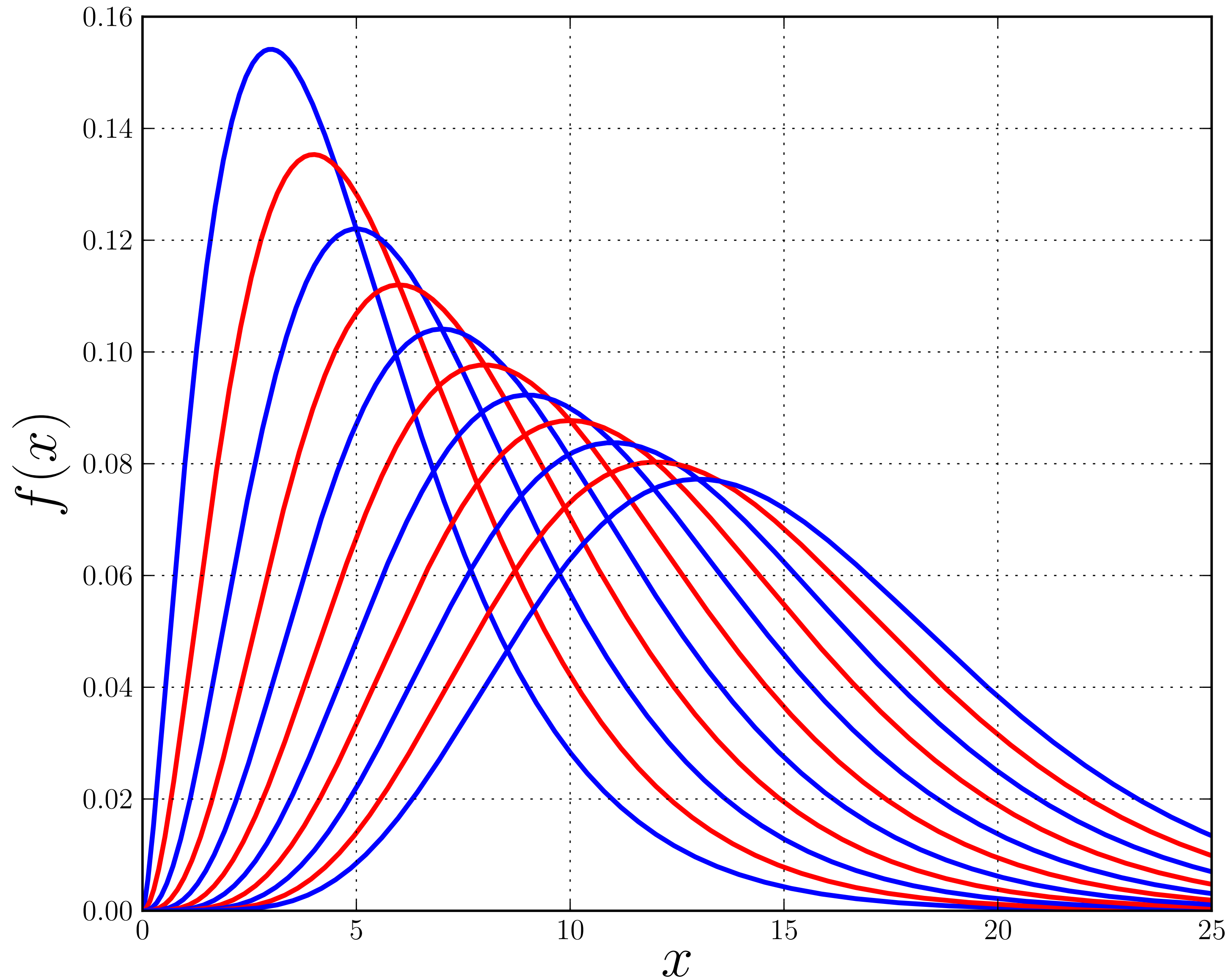
Suppose we test the lifetime of 16 bulbs. What is the probability that the sample standard deviation $\hat{S}$ satisfies $\hat{S} \geq \mathbf{129}$ hours ?

SOLUTION :

$$P(\hat{S} \geq 129) \quad = \quad P(\hat{S}^2 \geq 129^2) \quad = \quad P\Big(\frac{n-1}{\sigma^2}\,\hat{S}^2 \;\geq\; \frac{15}{100^2}\,129^2\Big)$$

$$\cong \; P(\chi^2_{15} \;\geq\; 24.96) \;\cong\; 5\,\% \qquad (\text{ from the } \chi^2 \text{ Table }) \;.$$

QUESTION : If $\hat{S} = 129$ then would you believe that $\sigma = 100$?



The Chi-Square *density* functions for $n = 5, 6, \cdots, 15$.

(For *large* n they look like *normal* density functions .)

EXERCISE :

In the preceding example, also compute

$$P(\ \chi^2_{15}\ \geq\ 24.96\ )$$

using the standard normal approximation .

EXERCISE :

Consider the same shipment of light bulbs :

- The lifetime of the bulbs has a *normal distribution* .
- The mean lifetime is not given.
- The *standard deviation* is claimed to be $\sigma = 100$ hours.

Suppose we test the lifetime of *only* 6 *bulbs* .

- For what value of s is $P(\hat{S} \leq s)\ =\ 5\ \%$?

EXAMPLE : For the data below from a *normal population* :

- Estimate the population standard deviation.
- Determine a 95 percent confidence interval for σ.

-0.047	0.126	-0.037	0.148
0.198	0.073	-0.025	-0.070
-0.197	-0.026	-0.062	-0.004
-0.164	0.265	-0.274	0.188

SOLUTION : We find (with $n = 16$) that

$$\bar{X} = \frac{1}{n} \sum_{1}^{n} X_i = 0.00575 ,$$

and

$$\hat{S}^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X})^2 = 0.02278 .$$

SOLUTION : We have $n = 16$, $\bar{X} = 0.00575$, $\hat{S}^2 = 0.02278$.

- Estimate the population standard deviation :

  ANSWER : $\sigma \cong \hat{S} = \sqrt{0.02278} = \mathbf{0.15095}$.

- Compute a 95 percent confidence interval for σ :

  ANSWER : From the *Chi-Square Table* :

$$P(\chi^2_{15} \leq 6.26) = 0.025 \quad , \quad P(\chi^2_{15} > 27.49) = 0.025 \ .$$

$$\frac{(n-1)\ \hat{S}^2}{\sigma^2} = 6.26 \quad \Rightarrow \quad \sigma^2 = \frac{(n-1)\hat{S}^2}{6.26} = \frac{15 \cdot 0.02278}{6.26} = 0.05458$$

$$\frac{(n-1)\ \hat{S}^2}{\sigma^2} = 27.49 \quad \Rightarrow \quad \sigma^2 = \frac{(n-1)\hat{S}^2}{27.49} = \frac{15 \cdot 0.02278}{27.49} = 0.01223$$

Thus the 95 % *confidence interval* for σ is

$$[\ \sqrt{0.01223}\ ,\ \sqrt{0.05458}\ ] = [\ \mathbf{0.106}\ ,\ \mathbf{0.234}\ ]\ .$$

Samples from Finite Populations

Samples from a *finite population* can be taken

(1) *with replacement*

(2) *without replacement*

- In Case 1 the sample

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ ,$$

  may contain the *same outcome* more than once.

- In Case 2 the outcomes are *distinct* .

- Case 2 arises, *e.g.*, when the experiment *destroys* the sample.

EXAMPLE :

Suppose a bag contains *three* balls, numbered 1, 2, and 3.

A *sample* of *two* balls is drawn at random from the bag.

Recall that (here with $n = 2$) :

$$\bar{X} \equiv \frac{1}{n} (X_1 + X_2 + \cdots + X_n) .$$

$$S^2 \equiv \frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2 .$$

For both, sampling *with* and *without replacement* , compute

$$E[\bar{X}] \quad \text{and} \quad E[S^2] .$$

- *With replacement* : The possible samples are

$$(1,1)\,,\ (1,2)\,,\ (1,3)\,,\ (2,1)\,,\ (2,2)\,,\ (2,3)\,,\ (3,1)\,,\ (3,2)\,,\ (3,3)\,,$$

each with equal probability $\frac{1}{9}$.

The *sample means* $\bar{X}$ are

$$1\ ,\ \frac{3}{2}\ ,\ 2\ ,\ \frac{3}{2}\ ,\ 2\ ,\ \frac{5}{2}\ ,\ 2\ ,\ \frac{5}{2}\ ,\ 3\ ,$$

with

$$E[\bar{X}] \ = \ \frac{1}{9}\,(1 + \frac{3}{2} + 2 + \frac{3}{2} + 2 + \frac{5}{2} + 2 + \frac{5}{2} + 3) \ = \ 2\ .$$

The *sample variances* S^2 are

$$0\ ,\ \frac{1}{4}\ ,\ 1\ ,\ \frac{1}{4}\ ,\ 0\ ,\ \frac{1}{4}\ ,\ 1\ ,\ \frac{1}{4}\ ,\ 0\ . \qquad (\text{ Check ! })$$

with

$$E[S^2] \ = \ \frac{1}{9}\,(0 + \frac{1}{4} + 1 + \frac{1}{4} + 0 + \frac{1}{4} + 1 + \frac{1}{4} + 0) \ = \ \frac{1}{3}\ .$$

- *Without replacement* : The possible samples are

$$(1,2)\ ,\ (1,3)\ ,\ (2,1)\ ,\ (2,3)\ ,\ (3,1)\ ,\ (3,2)\ ,$$

each with equal probability $\frac{1}{6}$.

The *sample means* $\bar{X}$ are

$$\frac{3}{2}\ ,\quad 2\ ,\quad \frac{3}{2}\ ,\quad \frac{5}{2}\ ,\quad 2\ ,\quad \frac{5}{2}\ ,$$

with *expected value*

$$E[\bar{X}] \ = \ \frac{1}{6}\,(\frac{3}{2} \ + \ 2 \ + \ \frac{3}{2} \ + \ \frac{5}{2} \ + \ 2 \ + \ \frac{5}{2}) \ = \ 2\ .$$

The *sample variances* S^2 are

$$\frac{1}{4}\ ,\quad 1\ ,\quad \frac{1}{4}\ ,\quad \frac{1}{4}\ ,\quad 1\ ,\quad \frac{1}{4}\ . \qquad (\text{ Check ! })$$

with *expected value*

$$E[S^2] \ = \ \frac{1}{6}\,(\frac{1}{4} \ + \ 1 \ + \ \frac{1}{4} \ + \ \frac{1}{4} \ + \ 1 \ + \ \frac{1}{4}) \ = \ \frac{1}{2}\ .$$

EXAMPLE : (continued $\cdots$)

A bag contains *three* balls, numbered 1, 2, and 3.

A *sample* of *two* balls is drawn at random from the bag.

We have computed $E[\bar{X}]$ and $E[S^2]$:

- *With* replacement : $E[\bar{X}] = 2$, $E[S^2] = \frac{1}{3}$,
- *Without* replacement : $E[\bar{X}] = 2$, $E[S^2] = \frac{1}{2}$.

We also know the *population mean* and *variance* :

$$\mu = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 2 ,$$

$$\sigma^2 = (1-2)^2 \cdot \frac{1}{3} + (2-2)^2 \cdot \frac{1}{3} + (3-2)^2 \cdot \frac{1}{3} = \frac{2}{3} .$$

EXAMPLE : (continued $\cdots$)

We have computed :

- *Population* statistics : $\mu = 2$, $\sigma^2 = \frac{2}{3}$,
- Sampling *with* replacement : $E[\bar{X}] = 2$, $E[S^2] = \mathbf{\frac{1}{3}}$,
- Sampling *without* replacement : $E[\bar{X}] = 2$, $E[S^2] = \frac{1}{2}$.

According to the earlier Theorem

$$E[S^2] = (1 - \frac{1}{n})\, \sigma^2 .$$

In this example the *sample size* is $n = 2$, thus

$$E[S^2] = (1 - \frac{1}{2})\, \sigma^2 = \mathbf{\frac{1}{3}} .$$

NOTE : $E[S^2]$ is *wrong* for sampling *without replacement* !

QUESTION :

Why is $E[S^2]$ *wrong* for sampling *without replacement* ?

ANSWER : Without replacement the outcomes X_k of a sample

$$X_1\ ,\ X_2\ ,\ \cdots\ ,\ X_n,$$

are *not independent* !

In our example , where $n = 2$, and where the possible samples are

$$(1,2)\ ,\ (1,3)\ ,\ (2,1)\ ,\ (2,3)\ ,\ (3,1)\ ,\ (3,2)\ ,$$

we have, *e.g.*,

$$P(X_2 = 1 \mid X_1 = 1) \ = \ 0 \quad , \quad P(X_2 = 1 \mid X_1 = 2) \ = \ \frac{1}{2}\ .$$

Thus X_1 and X_2 are *not independent* . (Why not ?)

NOTE :

Let N be the *population size* and n the *sample size* .

Suppose N is *very large* compared to n .

For example, $n = 2$, and the population is

$$\{ 1 \ , \ 2 \ , \ 3 \ , \ \cdots \ , \ N \} \ .$$

Then we still have

$$P(X_2 = 1 \mid X_1 = 1) \quad = \quad 0 \ ,$$

but for $k \neq 1$ we have

$$P(X_2 = k \mid X_1 = 1) \quad = \quad \frac{1}{N-1} \ .$$

One could say that X_1 and X_2 are "*almost independent*" . (Why ?)

The Sample Correlation Coefficient

Recall the *covariance* of random variables X and Y :

$$\sigma_{X,Y} \equiv Cov(X,Y) \equiv E[\,(X-\mu_X)\,(Y-\mu_Y)\,] = E[XY] - E[X]\,E[Y]\,.$$

It is often better to use a *scaled* version, the *correlation coefficient*

$$\rho_{X,Y} \equiv \frac{\sigma_{X,Y}}{\sigma_X\ \sigma_Y}\ ,$$

where σ_X and σ_Y are the standard deviation of X and Y .

We have

- $|\,\sigma_{X,Y}\,| \leq \sigma_X\ \sigma_Y$, (the *Cauchy-Schwartz inequality*)
- Thus $|\,\rho_{X,Y}\,| \leq 1$, (Why ?)
- If X and Y are independent then $\rho_{X,Y} = 0$. (Why ?)

Similarly, the *sample correlation coefficient* of a data set

$$\{ (X_i, Y_i) \}_{i=1}^{N} ,$$

is defined as

$$R_{X,Y} \quad \equiv \quad \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{N}(X_i - \bar{X})^2} \sqrt{\sum_{i=1}^{N}(Y_i - \bar{Y})^2}} ;$$

for which we have another version of the *Cauchy-Schwartz inequality* :

$$| R_{X,Y} | \quad \leq \quad 1 .$$

Like the covariance, $R_{X,Y}$ measures "*concordance*" of X and Y :

- If $X_i > \bar{X}$ when $Y_i > \bar{Y}$ and $X_i < \bar{X}$ when $Y_i < \bar{Y}$ then
$$R_{X,Y} > 0 .$$
- If $X_i > \bar{X}$ when $Y_i < \bar{Y}$ and $X_i < \bar{X}$ when $Y_i > \bar{Y}$ then
$$R_{X,Y} < 0 .$$

The *sample correlation coefficient*

$$R_{X,Y} \quad \equiv \quad \frac{\sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^N (X_i - \bar{X})^2} \sqrt{\sum_{i=1}^N (Y_i - \bar{Y})^2}} \; .$$

can also be used to *test for linearity* of the data.

In fact,

- If $| \, R_{X,Y} \, | \; = \; 1$ then X and Y are related *linearly* .

Specifically,

- If $R_{X,Y} = \; 1$ then $Y_i = cX_i + d$, for constants c, d, with $c > 0$.
- If $R_{X,Y} = -1$ then $Y_i = cX_i + d$, for constants c, d, with $c < 0$.

Also,

- If $| \, R_{X,Y} \, | \; \cong \; 1$ then X and Y are *almost linear* .

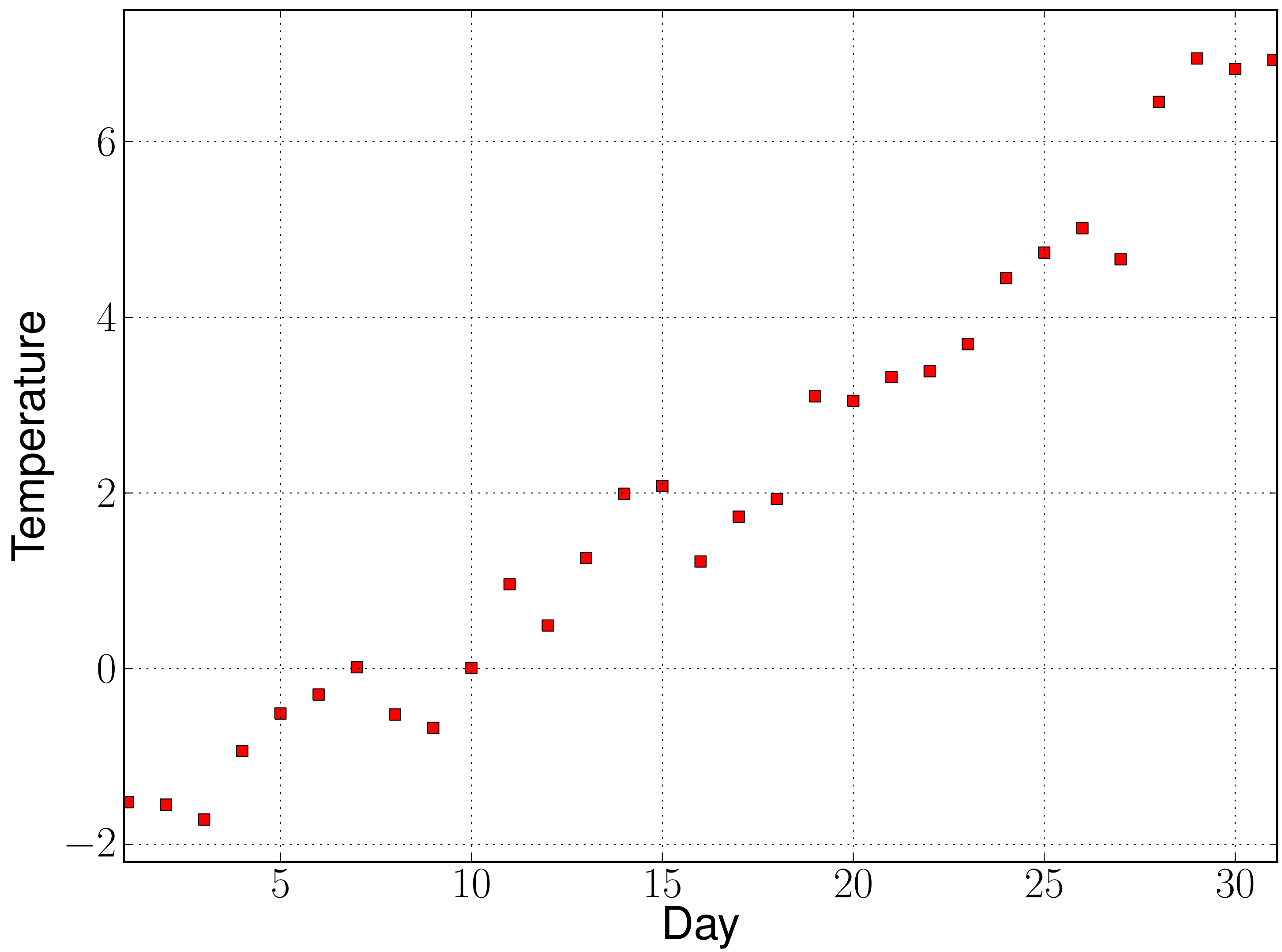
EXAMPLE :

- Consider the *average daily high temperature* in Montreal in March.
- The Table shows these averages, taken over a number of years :

1	-1.52	8	-0.52	15	2.08	22	3.39	29	6.95
2	-1.55	9	-0.67	16	1.22	23	3.69	30	6.83
3	-1.72	10	0.01	17	1.73	24	4.45	31	6.93
4	-0.94	11	0.96	18	1.93	25	4.74		
5	-0.51	12	0.49	19	3.10	26	5.01		
6	-0.29	13	1.26	20	3.05	27	4.66		
7	0.02	14	1.99	21	3.32	28	6.45		

Average daily high temperature in Montreal in March : 1943-2014 .
(Source : `http://climate.weather.gc.ca/`)

These data have *sample correlation coefficient* $R_{X,Y} = \mathbf{0.98}$.



A *scatter diagram* showing the average daily high temperature.

The sample correlation coefficient is $R_{X,Y} = \mathbf{0.98}$

EXERCISE :

- The Table below shows class attendance and course grade/100.
- The attendance was sampled in 18 sessions.

11	47	13	43	15	70	17	72	18	96	14	61	5	25	17	74
16	85	13	82	16	67	17	91	16	71	16	50	14	77	12	68
8	62	13	71	12	56	15	81	16	69	18	93	18	77	17	48
14	82	17	66	16	91	17	67	7	43	15	86	18	85	17	84
11	43	17	66	18	57	18	74	13	73	15	74	18	73	17	71
14	69	15	85	17	79	18	84	17	70	15	55	14	75	15	61
16	61	4	46	18	70	0	29	17	82	18	82	16	82	14	68
9	84	15	91	15	77	16	75								

Class attendance - Course grade

- Draw a *scatter diagram* showing the data.
- Determine the *sample correlation coefficient* .
- Any *conclusions* ?

Maximum Likelihood Estimators

EXAMPLE :

Suppose a random variable has a *normal distribution with mean* 0 .

Thus the density function is

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}x^2/\sigma^2} \ .$$

- Suppose we don't know σ (the *population standard deviation*).
- How can we *estimate* σ from observed data ?
- (We want a *formula* for estimating σ .)
- Don't we already have such a formula ?

EXAMPLE : (continued $\cdots$)

We know we can *estimate* σ^2 by the *sample variance*

$$S^2 \quad \equiv \quad \frac{1}{n} \sum_{k=1}^{n} (X_k - \bar{X})^2 \ .$$

In fact, we have proved that

$$E[S^2] \quad = \quad (1 - \frac{1}{n}) \ \sigma^2 \ .$$

- Thus, we can call S^2 an *estimator* of σ^2 .

- The "maximum likelihood procedure" *derives* such estimators.

The *maximum likelihood procedure* is the following :

Let

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ ,$$

be

independent, identically distributed ,

each having

density function $f(x \ ; \ \sigma)$,

with *unknown parameter* σ .

By *independence* , the *joint density function* is

$$f(x_1, x_2, \ \cdots \ , x_n \ ; \ \sigma) \ = \ f(x_1;\sigma) \ f(x_2;\sigma) \ \cdots \ f(x_n;\sigma) \ ,$$

DEFINITION : The *maximum likelihood estimate* $\hat{\sigma}$ is

the value of σ that *maximizes* $f(x_1, x_2, \ \cdots \ , x_n \ ; \ \sigma)$.

NOTE : $\hat{\sigma}$ will be a *function* of $x_1, x_2, \ \cdots \ , x_n$.

EXAMPLE : For our *normal distribution* with mean 0 we have

$$f(x_1, x_2, \cdots, x_n \; ; \; \sigma) \;=\; \frac{e^{-\frac{1}{2\sigma^2}\sum_{k=1}^n x_k^2}}{(\sqrt{2\pi}\;\sigma)^n} \; . \qquad (\text{ Why ? })$$

To find the maximum (with respect to σ) we set

$$\frac{d}{d\sigma}\, f(x_1, x_2, \cdots, x_n \; ; \; \sigma) \;=\; 0 \; , \qquad (\text{ by Calculus ! })$$

or, *equivalently* , we set

$$\frac{d}{d\sigma}\, \log\Big(\frac{e^{-\frac{1}{2\sigma^2}\sum_{k=1}^n x_k^2}}{\sigma^n}\Big) \;=\; 0 \; . \qquad (\text{ Why } \textit{equivalent} \text{ ? })$$

Taking the (natural) logarithm gives

$$\frac{d}{d\sigma}\,\Big(-\frac{1}{2\sigma^2}\sum_{k=1}^n x_k^2 \;-\; n \log \sigma \Big) \;=\; 0 \; .$$

EXAMPLE : (continued $\cdots$)

We had

$$\frac{d}{d\sigma} \Big(-\frac{1}{2\sigma^2} \sum_{k=1}^{n} x_k^2 \ - \ n \log \sigma \Big) \ = \ 0 \ .$$

Taking the derivative gives

$$\frac{\sum_{k=1}^{n} x_k^2}{\sigma^3} \ - \ \frac{n}{\sigma} \ = \ 0 \ ,$$

from which

$$\hat{\sigma}^2 \ = \ \frac{1}{n} \sum_{k=1}^{n} x_k^2 \ .$$

Thus we have derived the *maximum likelihood estimate*

$$\hat{\sigma} \ = \ \frac{1}{\sqrt{n}} \Big(\sum_{k=1}^{n} X_k^2 \Big)^{\frac{1}{2}} \ . \qquad \text{(Surprise ?)}$$

EXERCISE :

Suppose a random variable has the *general normal density function*

$$f(x\ ;\ \mu,\sigma) \quad = \quad \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\ ,$$

with *unknown* mean μ *and unknown* standard deviation σ .

Derive *maximum likelihood estimators* for *both* μ and σ as follows :

For the *joint density function*

$$f(x_1, x_2, \cdots, x_n; \mu, \sigma)\ =\ f(x_1; \mu, \sigma)\ f(x_2; \mu, \sigma) \cdots f(x_n; \mu, \sigma)\ ,$$

- Take the log of $f(x_1, x_2,\ \cdots\ , x_n\ ;\ \mu, \sigma)$.
- Set the *partial derivative* w.r.t. μ equal to zero.
- Set the *partial derivative* w.r.t. σ equal to zero.
- Solve these two equations for $\hat{\mu}$ and $\hat{\sigma}$.

EXERCISE : (continued $\cdots$)

The *maximum likelihood estimators* turn out to be

$$\hat{\mu} \;=\; \frac{1}{n} \sum_{k=1}^{n} X_k \; ,$$

$$\hat{\sigma} \;=\; \frac{1}{\sqrt{n}} \Big(\sum_{k=1}^{n} (X_k - \bar{X})^2 \Big)^{\frac{1}{2}} \; ,$$

that is,

$$\hat{\mu} \;=\; \bar{X} \; , \qquad \text{(the } \textit{sample mean} \text{)} \; ,$$

$$\hat{\sigma} \;=\; S \quad \text{(the } \textit{sample standard deviation} \text{)} \; .$$

NOTE :

- Earlier we *defined* the *sample variance* as

$$S^2 \quad = \quad \frac{1}{n} \sum_{k=1}^{n} \, (X_k - \bar{X})^2 \ .$$

- Then we proved that, in general,

$$E[S^2] \quad = \quad (1 - \frac{1}{n}) \, \sigma^2 \quad \cong \quad \sigma^2.$$

- In the preceding exercise we *derived* the estimator for σ !

- (But we did so *specifically* for the general normal distribution.)

EXERCISE :

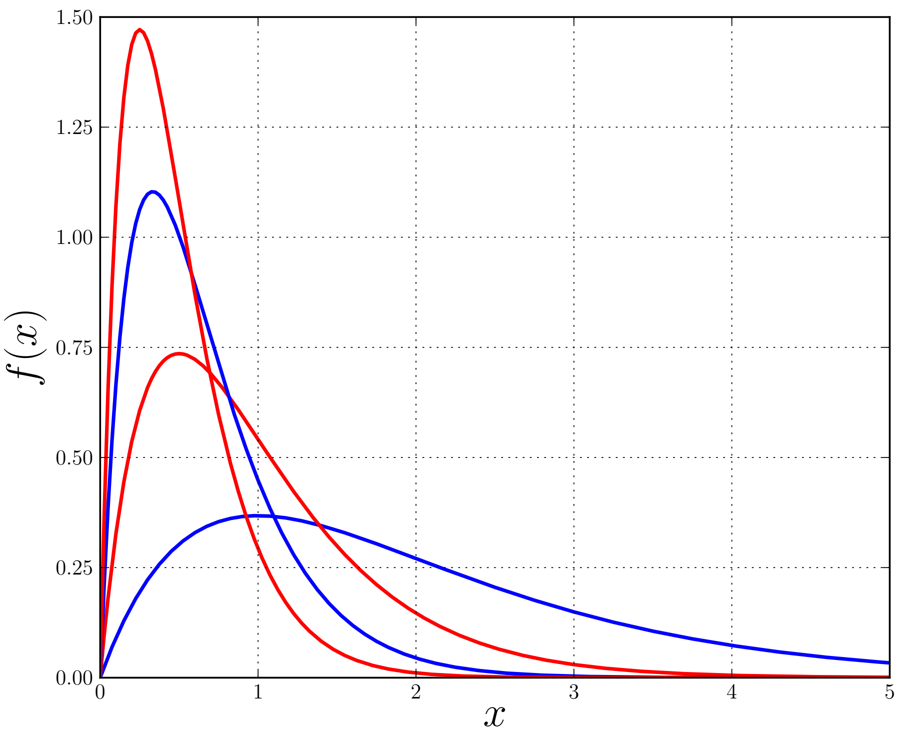
A random variable has the standard *exponential distribution* with density function

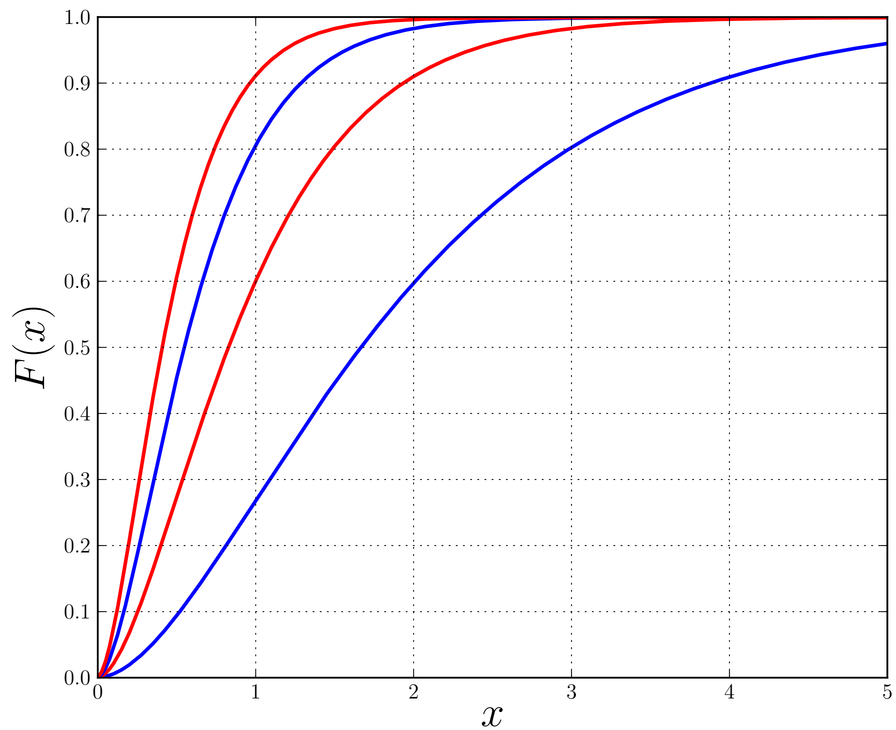
$$f(x\ ;\ \lambda) \ = \ \begin{cases} \lambda e^{-\lambda x}\ , & x > 0 \\ \\ 0\ , & x \leq 0 \end{cases}$$

- Suppose we don't know λ .
- Derive the *maximum likelihood estimator* of λ .
- (Can you guess what the formula will be ?)

EXAMPLE : Consider the *special* exponential density function

$$f(x\ ;\ \lambda) \ = \ \begin{cases} \lambda^2 x e^{-\lambda x}\ , & x > 0 \\ \\ 0\ , & x \le 0 \end{cases}$$





Density and distribution functions for $\lambda = 1\ ,\ 2\ ,\ 3\ ,\ 4$.

EXAMPLE : (continued $\cdots$)

For the *maximum likelihood estimator* of λ , we have

$$f(x\ ;\ \lambda) \quad = \quad \lambda^2\ x\ e^{-\lambda x}\ , \qquad \text{for } x > 0\ ,$$

so, assuming independence, the *joint density function* is

$$f(x_1, x_2,\ \cdots\ , x_n\ ;\ \lambda) \quad = \quad \lambda^{2n}\ x_1 x_2\ \cdots\ x_n\ e^{-\lambda(x_1+x_2+\ \cdots\ +x_n)}\ .$$

To find the *maximum* (with respect to λ) we set

$$\frac{d}{d\lambda} \log\Big(\lambda^{2n}\ \ x_1 x_2\ \cdots\ x_n\ \ e^{-\lambda(x_1+x_2+\ \cdots\ +x_n)}\Big) \quad = \quad 0\ .$$

Taking the logarithm gives

$$\frac{d}{d\lambda}\Big(2n\ \log\lambda\ \ +\ \ \sum_{k=1}^{n} \log x_k\ \ -\ \ \lambda\sum_{k=1}^{n}\ x_k\Big) \ = \ 0\ .$$

EXAMPLE : (continued $\cdots$)

We had

$$\frac{d}{d\lambda} \Big(2n \log \lambda \ + \ \sum_{k=1}^{n} \log x_k \ - \ \lambda \sum_{k=1}^{n} x_k \Big) \ = \ 0 \ .$$

Differentiating gives

$$\frac{2n}{\lambda} \ - \ \sum_{k=1}^{n} x_k \ = \ 0 \ ,$$

from which

$$\hat{\lambda} \ = \ \frac{2n}{\sum_{k=1}^{n} x_k} \ .$$

Thus we have derived the *maximum likelihood estimate*

$$\hat{\lambda} \ = \ \frac{2n}{\sum_{k=1}^{n} X_k} \ = \ \frac{2}{\bar{X}} \ .$$

NOTE : This result suggests that perhaps $E[X] \ = \ 2/\lambda$. (Why ?)

EXERCISE :

For the special exponential density function in the preceding example,

$$f(x\ ;\ \lambda) \ = \ \begin{cases} \lambda^2 x e^{-\lambda x}\ , & x > 0 \\ \\ 0\ , & x \leq 0 \end{cases}$$

- Verify that

$$\int_0^\infty f(x\ ;\ \lambda)\ dx \ = \ 1\ .$$

- Also compute

$$E[X] \ = \ \int_0^\infty x\ f(x\ ;\ \lambda)\ dx$$

- Is it indeed true that

$$E[X] \ = \ \frac{2}{\lambda} \qquad ?$$

NOTE :

- Maximum likelihood estimates also work in the *discrete case* .
- In such case we maximize the *probability mass function* .

EXAMPLE :

Find the maximum likelihood estimator of p in the *Bernoulli trial*

$$P(X=1) \quad = \quad p \ ,$$
$$P(X=0) \quad = \quad 1-p \ .$$

SOLUTION : We can write

$$P(x\ ;\ p) \quad \equiv \quad P(X=x) \quad = \quad p^x\ (1-p)^{1-x}\ , \qquad (\ x=0,1\ ) \qquad (\ !\ )$$

so, assuming *independence* , the *joint probability mass function* is

$$P(x_1,x_2,\ \cdots\ ,x_n;p) \ = \ p^{x_1}(1-p)^{1-x_1}\ p^{x_2}(1-p)^{1-x_2}\ \cdots\ p^{x_n}(1-p)^{1-x_n}$$

$$= \ p^{\sum_{k=1}^n x_k} \ \cdot \ (1-p)^n \ \cdot \ (1-p)^{-\sum_{k=1}^n x_k} \ .$$

EXAMPLE : (continued $\cdots$)

We found

$$P(x_1, x_2, \quad \cdots \quad , x_n; p) \quad = \quad p^{\sum_{k=1}^n x_k} \quad \cdot \quad (1-p)^n \quad \cdot \quad (1-p)^{-\sum_{k=1}^n x_k} \quad .$$

To find the *maximum* (with respect to p) we set

$$\frac{d}{dp} \log\Big(p^{\sum_{k=1}^n x_k} \quad \cdot \quad (1-p)^n \quad \cdot \quad (1-p)^{-\sum_{k=1}^n x_k} \Big) \quad = \quad 0 \ .$$

Taking the logarithm gives

$$\frac{d}{dp} \Big(\log\, p \sum_{k=1}^n x_k \quad + \quad n \log(1-p) \quad - \quad \log\,(1-p) \sum_{k=1}^n x_k \Big) \quad = \quad 0 \ .$$

Differentiating gives

$$\frac{1}{p} \sum_{k=1}^n x_k \quad - \quad \frac{n}{1-p} \quad + \quad \frac{1}{1-p} \sum_{k=1}^n x_k \quad = \quad 0 \ .$$

EXAMPLE : (continued $\cdots$)

We found

$$\frac{1}{p}\sum_{k=1}^{n} x_k \quad - \quad \frac{n}{1-p} \quad + \quad \frac{1}{1-p}\sum_{k=1}^{n} x_k \quad = \quad 0 \ ,$$

from which

$$\Big(\frac{1}{p} \ + \ \frac{1}{1-p} \Big) \sum_{k=1}^{n} x_k \quad = \quad \frac{n}{1-p} \ .$$

Multiplying by $1-p$ gives

$$\Big(\frac{1-p}{p} + 1 \Big) \ \sum_{k=1}^{n} x_k \quad = \quad \frac{1}{p} \ \sum_{k=1}^{n} x_k \quad = \quad n \ ,$$

from which we obtain the *maximum likelihood estimator*

$$\hat{p} \quad = \quad \frac{\sum_{k=1}^{n} X_k}{n} \quad \equiv \quad \bar{X} \ . \qquad\qquad (\text{ Surprise ? })$$

EXERCISE :

Consider the *Binomial* probability mass function

$$P(x\ ;\ p) \quad \equiv \quad P(X = x) \quad = \quad \binom{N}{x} \cdot p^x \cdot (1-p)^{N-x} \ ,$$

where x is an integer, $(0 \leq x \leq N)$.

- What is the *joint probability mass function* $P(x_1, x_2, \cdots, x_n; p)$?
- (Be sure to distinguish between N and n !)
- Determine the *maximum likelihood estimator* $\hat{p}$ of p .
- (Can you guess what $\hat{p}$ will be ?)

Hypothesis Testing

- Often we want to decide whether a *hypothesis* is True or False.

- To do so we gather data, *i.e.*, a *sample* .

- A typical hypothesis is that a random variable *has a given mean* .

- Based on the data we want to *accept* or *reject* the hypothesis.

- To illustrate concepts we consider an example in detail.

EXAMPLE :

We consider ordering a large shipment of 50 watt light bulbs.

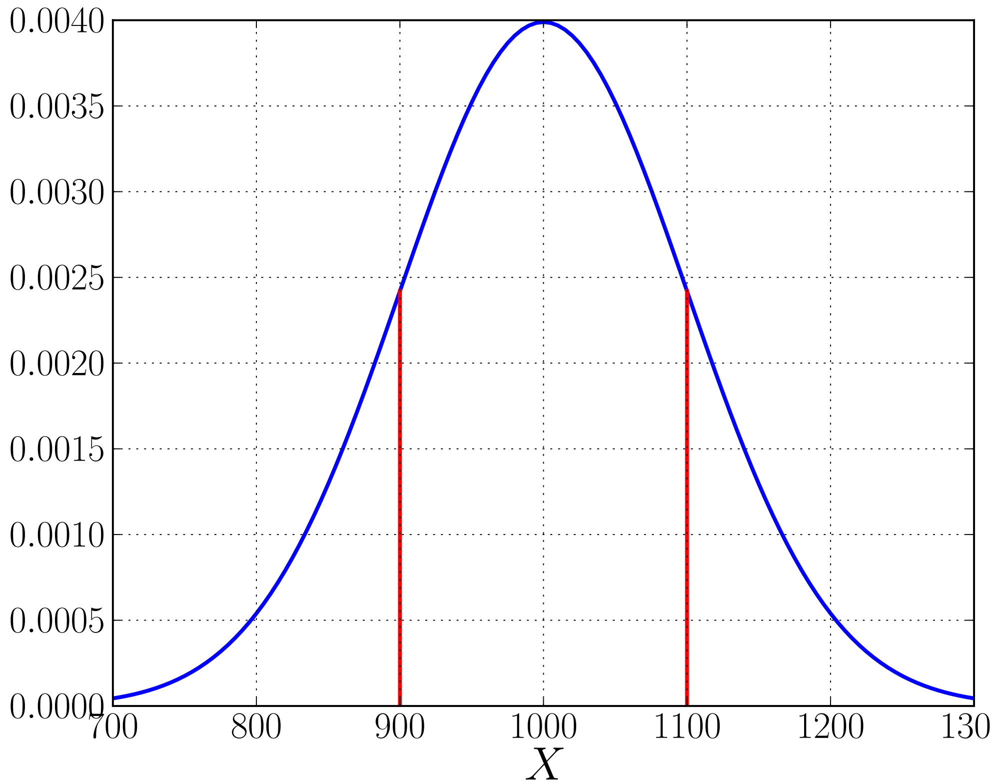
The manufacturer *claims* that :

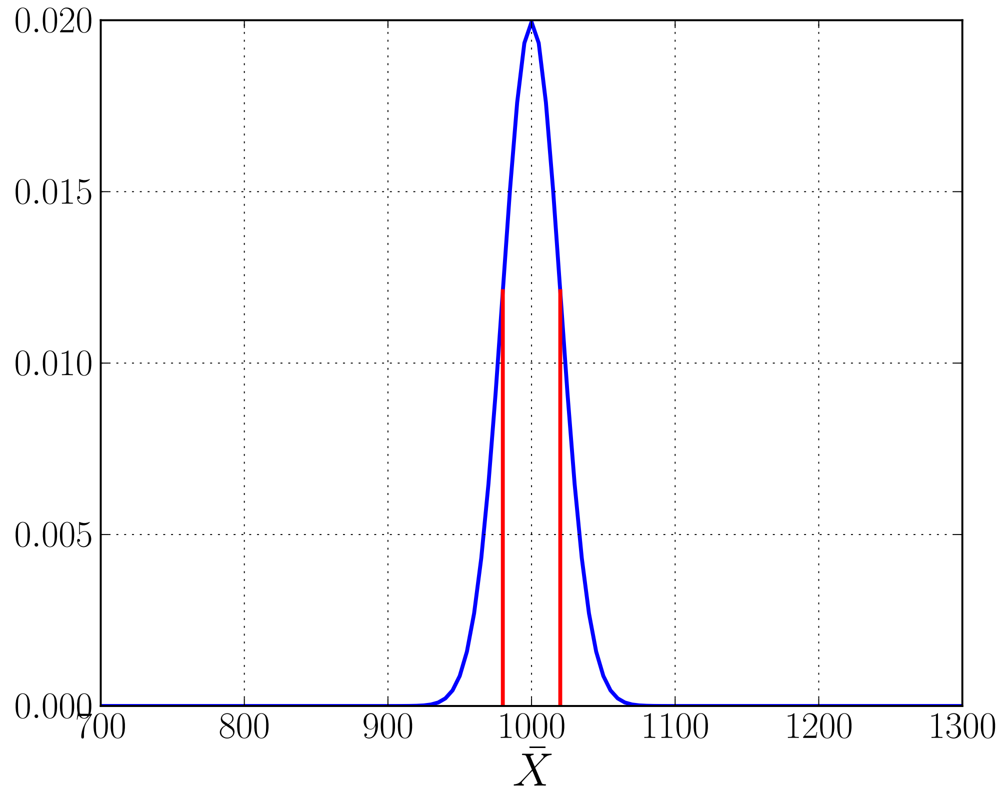
- The lifetime of the bulbs has a *normal distribution* .
- The mean lifetime is $\mu = 1000$ hours.
- The standard deviation is $\sigma = 100$ hours.

We want to *test the hypothesis that* $\mu = 1000$.

We *assume* that :

- The lifetime of the bulbs has indeed a *normal distribution* .
- The standard deviation is indeed $\sigma = 100$ hours.
- We *test* the lifetime of a sample of 25 *bulbs* .





Density function of X ,
also indicating $\mu \pm \sigma_X$,
$(\mu_X = 1000 \ , \ \ \sigma_X = 100)$.

Density function of $\bar{X}$ $(n = 25)$,
also indicating $\mu_{\bar{X}} \pm \sigma_{\bar{X}}$,
$(\mu_{\bar{X}} = 1000 \ , \ \ \sigma_{\bar{X}} = 20 \ )$.

EXAMPLE : (continued $\cdots$)

We *test* a *sample* of 25 light bulbs :

- We find the *sample* average lifetime is $\bar{X} = 960$ hours.

- Do we *accept* the hypothesis that $\mu = 1000$ hours ?

Using the standard normal Table we have the *one-sided probability*

$$P(\bar{X} \leq 960) \quad = \quad \Phi\Big(\frac{960 - 1000}{100/\sqrt{25}}\Big) \quad = \quad \Phi(-2.0) \quad = \quad 2.28\ \% \ ,$$

(assuming that the average lifetime is indeed 1000 hours).

- Would you *accept* the hypothesis that $\mu = 1000$?

- Would you *accept* (and pay for !) the shipment ?

EXAMPLE : (continued $\cdots$)

We *test* a *sample* of 25 light bulbs :

- Suppose instead the *sample* average lifetime is $\bar{X} = 1040$ hours.

- Do we *accept* that $\mu = 1000$ hours ?

Using the standard normal Table we have *one-sided probability*

$$P(\bar{X} \geq 1040) \ = \ 1 - \Phi\Big(\frac{1040 - 1000}{100/\sqrt{25}}\Big) \ = \ 1 - \Phi(2) \ = \ \Phi(-2) \ = \ 2.28\,\% \ ,$$

(assuming again that the average lifetime is indeed 1000 hours).

- Would you *accept the hypothesis* that that the mean is 1000 hours ?

- Would you *accept* the shipment ? (!)

EXAMPLE : (continued $\cdots$)

Suppose that we *accept the hypothesis* that $\mu = 1000$ if

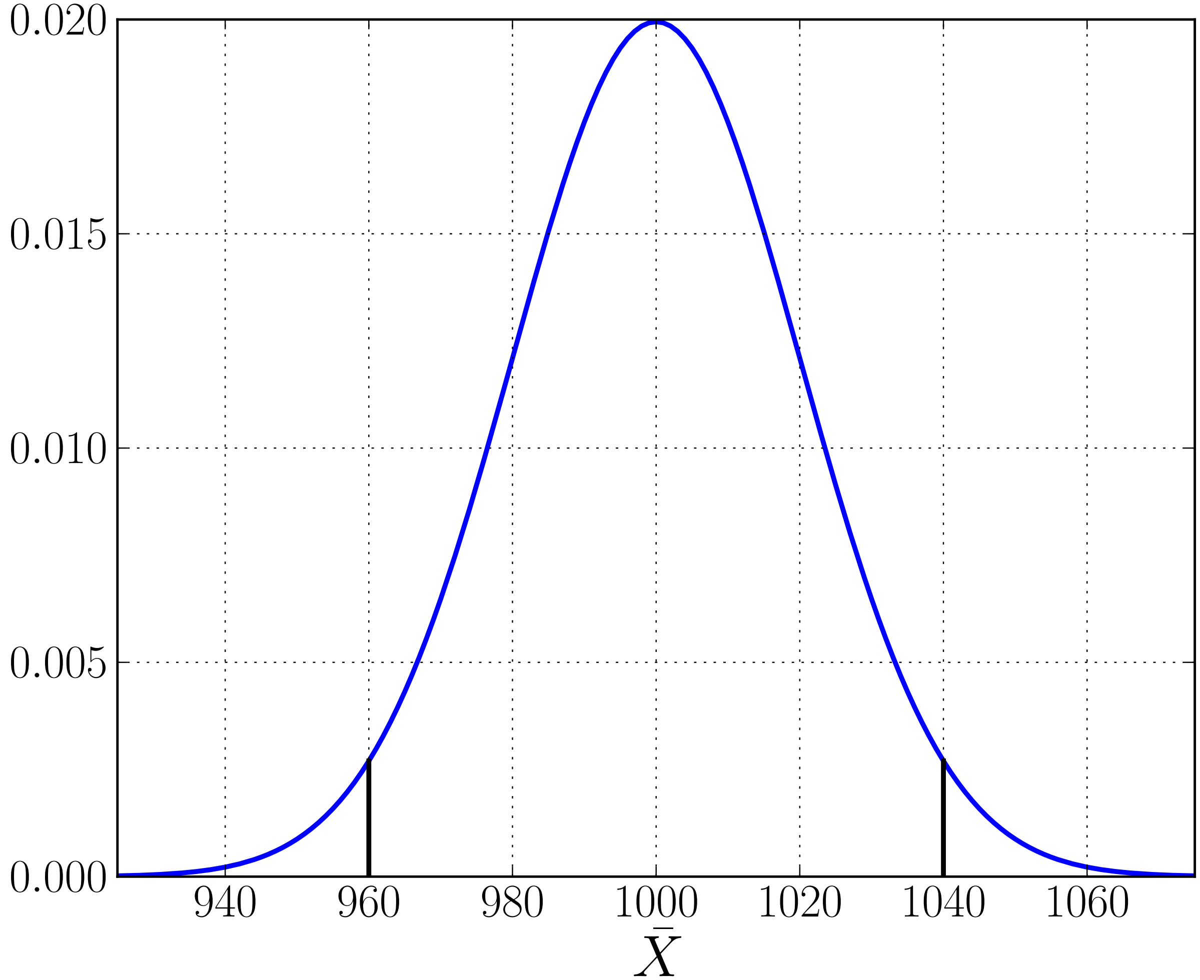
$$960 \ \leq \ \bar{X} \ \leq \ 1040 \ .$$

Thus, if indeed $\mu = 1000$, we *accept* the hypothesis with probability

$$P(\ | \bar{X} - 1000 | \leq 40 \ ) \ = \ 1 - 2\Phi\Big(\frac{960 - 1000}{100/\sqrt{25}}\Big) \ = \ 1 - 2\Phi(-2) \ \cong \ 95 \, \% \ ,$$

and we *reject* the hypothesis with probability

$$P(\ | \bar{X} - 1000 | \ \geq \ 40 \ ) \ = \ 100 \, \% \ - \ 95 \, \% \ = \ 5 \, \% \ .$$



Density function of $\bar{X}$ $(n = 25)$, with $\mu = \mu_{\bar{X}} = 1000$, $\sigma_{\bar{X}} = 20$,

$$P(960 \leq \bar{X} \leq 1040) \cong 95\%$$

EXAMPLE : (continued $\cdots$)

What is the probability of

acceptance of the hypothesis if μ is different from 1000 ?

- If the *actual mean* is $\mu = 980$, then acceptance has probability

$$P(960 \ \leq \ \bar{X} \ \leq \ 1040) \ = \ \Phi(\frac{1040-980}{100/\sqrt{25}}) \ - \ \Phi(\frac{960-980}{100/\sqrt{25}})$$

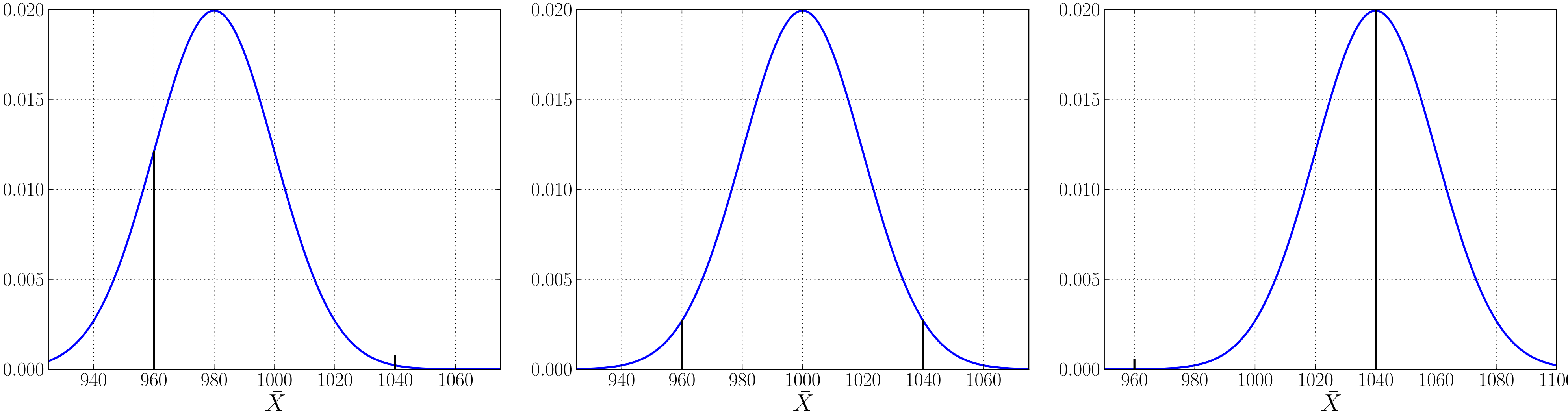
$$= \ \Phi(3) - \Phi(-1) \ = \ 1 - \Phi(-3) - \Phi(-1)$$

$$(1 - 0.0013) \ - \ 0.1587 \ = \ 84 \ \% \ .$$

- If the *actual mean* is $\mu = 1040$, then acceptance has probability

$$P(960 \ \leq \ \bar{X} \ \leq \ 1040) \ = \ \Phi(\frac{1040-1040}{100/\sqrt{25}}) \ - \ \Phi(\frac{960-1040}{100/\sqrt{25}})$$

$$= \ \Phi(0) - \Phi(-4) \ \cong \ 50 \ \% \ .$$



$$\mu = \mu_{\bar{X}} = 980 \qquad \mu = \mu_{\bar{X}} = 1000 \qquad \mu = \mu_{\bar{X}} = 1040$$

$$P(accept) = 84\% \qquad P(accept) = 95\% \qquad P(accept) = 50\%$$

Density functions of $\bar{X}$: $n = 25$, $\sigma_{\bar{X}} = 20$

QUESTION 1 : How does $P(accept)$ change when we "slide" the density function of $\bar{X}$ along the $\bar{X}$-axis , *i.e.*, when μ changes ?

QUESTION 2 : What is the effect of increasing the *sample size* n ?

EXAMPLE :

Now suppose there are *two lots* of light bulbs :

- Lot 1 : Light bulbs with mean life time $\mu_1 = 1000$ hours,

- Lot 2 : Light bulbs with mean life time $\mu_2 = 1100$ hours.

We want to *decide* which lot our sample of 25 bulbs is from.

Consider the *decision criterion* $\hat{x}$, where $1000 \leq \hat{x} \leq 1100$:

- If $\bar{X} \leq \hat{x}$ then the sample is from Lot 1 .

- If $\bar{X} > \hat{x}$ then the sample is from Lot 2 .

There are *two hypotheses* :

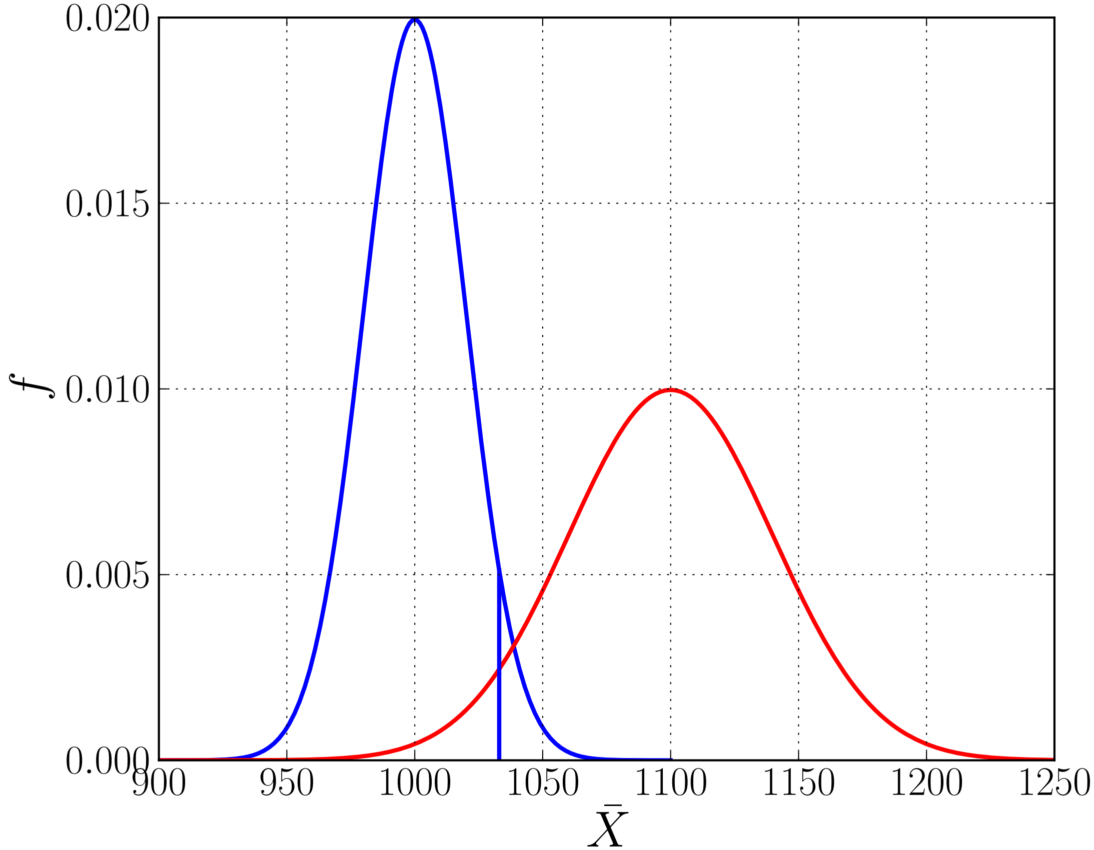
- H1 : The sample is from Lot 1 ($\mu_1 = 1000$) .
- H2 : The sample is from Lot 2 ($\mu_2 = 1100$) .

We can make *two types of errors* :

- Type 1 error : Accept H2 when H1 is True ,
- Type 2 error : Accept H1 when H2 is True ,

which happen when, for given *decision criterion* $\hat{x}$,

- Type 1 error : If $\bar{X} > \hat{x}$ and the sample is from Lot 1 .
- Type 2 error : If $\bar{X} \leq \hat{x}$ and the sample is from Lot 2 .



The density functions of $\bar{X}$ $(n = 25)$, also indicating $\hat{x}$.

blue : $(\mu_1, \sigma_1) = (1000, 100)$, *red* : $(\mu_2, \sigma_2) = (1100, 200)$.

Type 1 error : area under the *blue* curve, to the right of $\hat{x}$.

Type 2 error : area under the *red* curve, to the left of $\hat{x}$.

QUESTION : What is the effect of moving $\hat{x}$ on these errors ?

RECALL :

- Type 1 error : If $\bar{X} > \hat{x}$ and the sample is from Lot 1 .

- Type 2 error : If $\bar{X} \leq \hat{x}$ and the sample is from Lot 2 .

These errors occur with *probability*

- Type 1 error : $P(\bar{X} \geq \hat{x} \mid \mu = \mu_1 \equiv 1000)$.

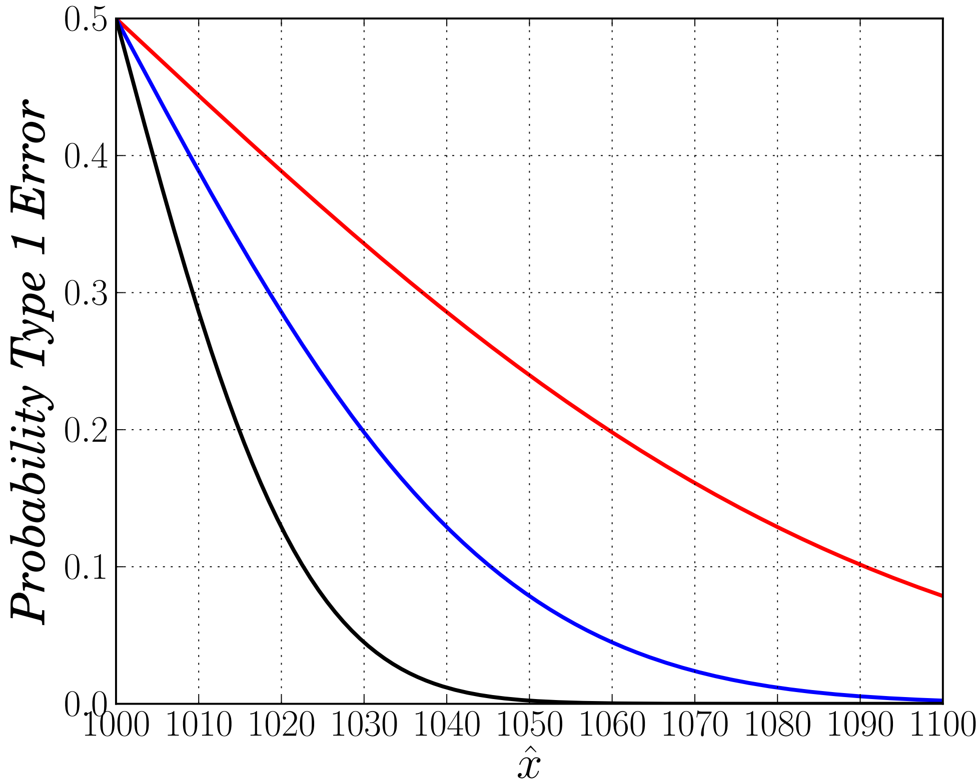
- Type 2 error : $P(\bar{X} \leq \hat{x} \mid \mu = \mu_2 \equiv 1100)$.

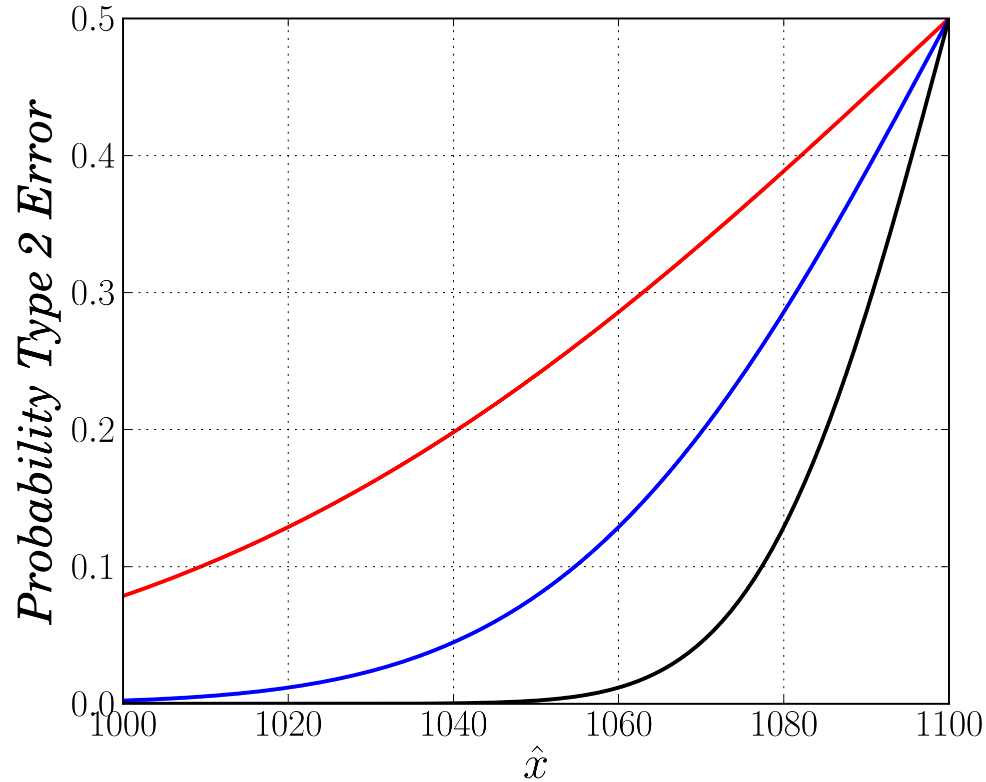
We should have, for the (rather bad) choice $\hat{x} = 1000$,

- Type 1 error : $P(\bar{X} \geq 1000 \mid \mu = \mu_1 \equiv 1000) = 0.5$.

and for the (equally bad) choice $\hat{x} = 1100$,

- Type 2 error : $P(\bar{X} \leq 1100 \mid \mu = \mu_2 \equiv 1100) = 0.5$.





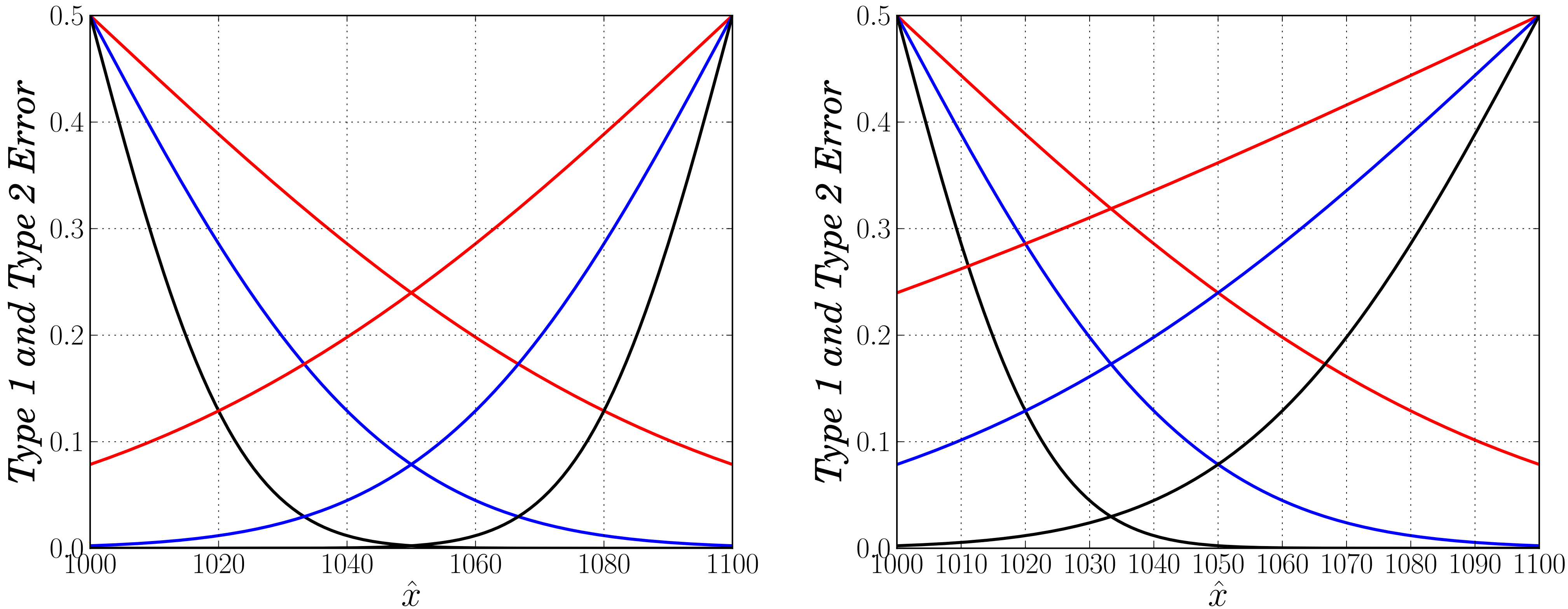
Probability of Type 1 error vs. $\hat{x}$

$(\mu_1, \sigma_1) = (1000, 100)$

Probability of Type 2 error vs. $\hat{x}$

$(\mu_2, \sigma_2) = (1100, 100)$.

Sample sizes : 2 (*red*) , 8 (*blue*) , 32 (*black*) .



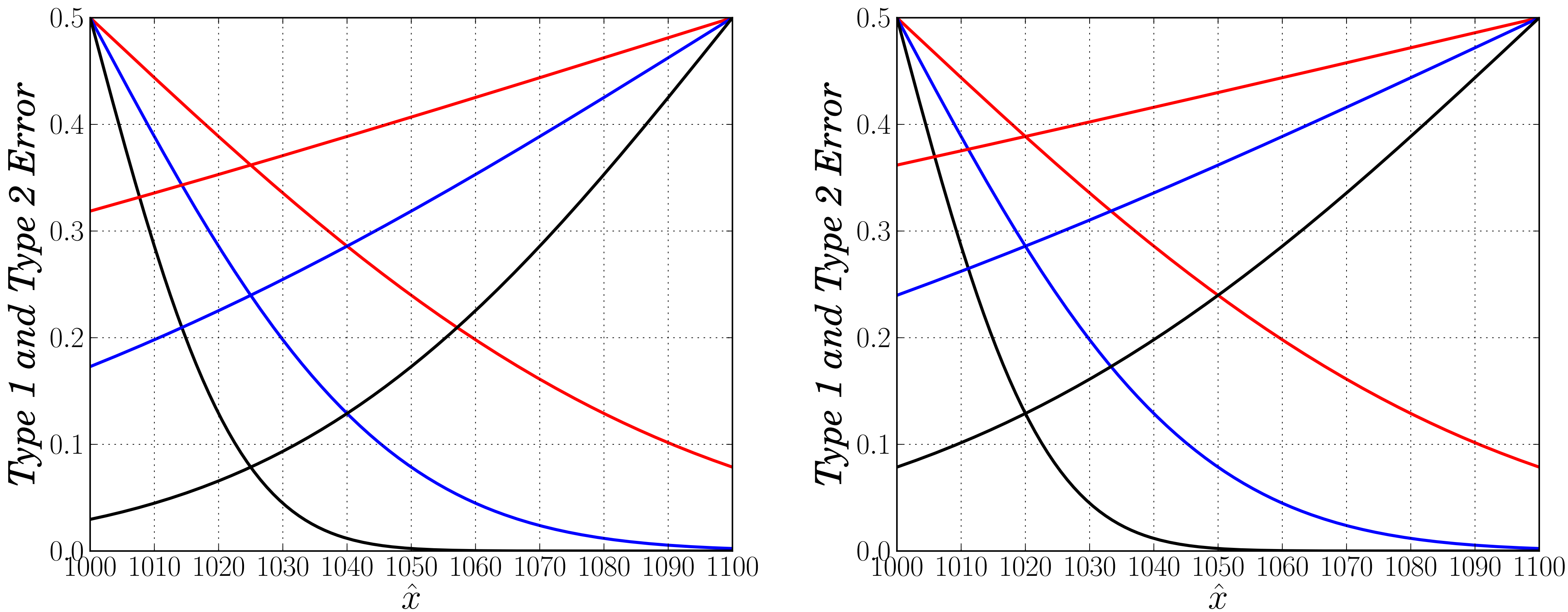
The probability of Type 1 and Type 2 errors versus $\hat{x}$.

Left : $(\mu_1, \boldsymbol{\sigma}_1) = (1000, \mathbf{100})$, $(\mu_2, \boldsymbol{\sigma}_2) = (1100, \mathbf{100})$.

Right : $(\mu_1, \boldsymbol{\sigma}_1) = (1000, \mathbf{100})$, $(\mu_2, \boldsymbol{\sigma}_2) = (1100, \mathbf{200})$.

Colors indicate sample size : 2 (*red*), 8 (*blue*), 32 (*black*) .

Curves of a given color intersect at the *minimax* $\hat{x}$-value.



The probability of Type 1 and Type 2 errors versus $\hat{x}$.

Left : $(\mu_1, \boldsymbol{\sigma}_1) = (1000, \mathbf{100})$, $(\mu_2, \boldsymbol{\sigma}_2) = (1100, \mathbf{300})$.

Right : $(\mu_1, \boldsymbol{\sigma}_1) = (1000, \mathbf{100})$, $(\mu_2, \boldsymbol{\sigma}_2) = (1100, \mathbf{400})$.

Colors indicate sample size : 2 (*red*), 8 (*blue*), 32 (*black*) .

Curves of a given color intersect at the *minimax* $\hat{x}$-value.

NOTE :

- There is an *optimal* value $\hat{x}^*$ of $\hat{x}$.

- At $\hat{x}^*$ the value of

$$\max \ \{ \ P(\text{Type 1 Error}) \ , \ P(\text{Type 2 Error}) \ \}$$

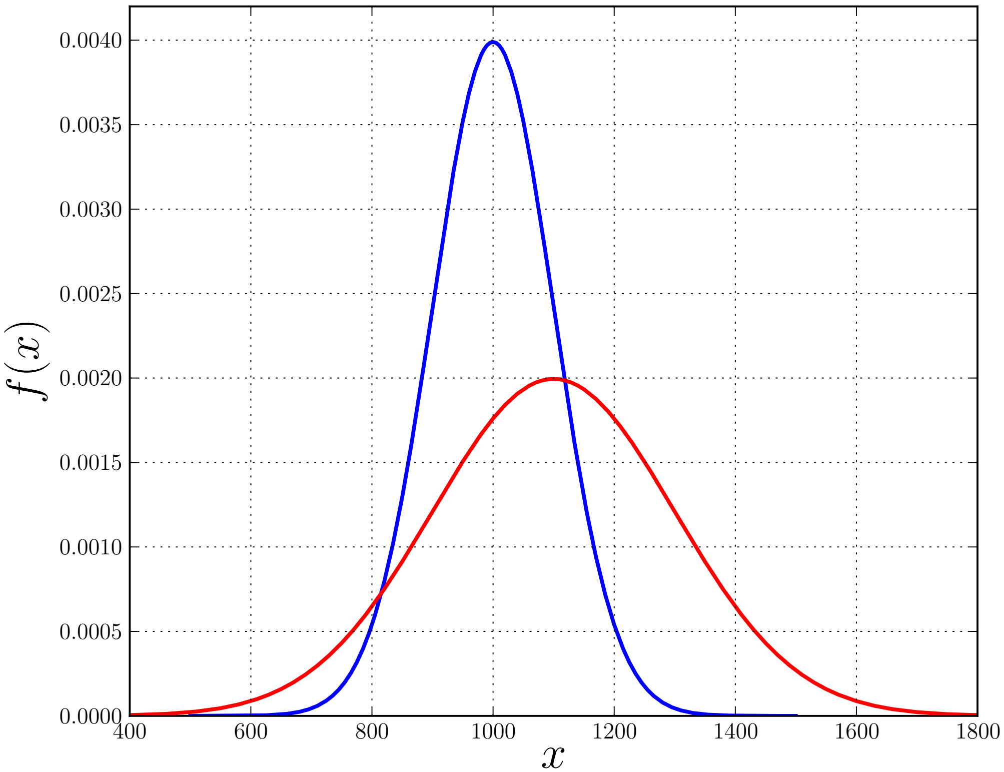
  is *minimized* .

- We call $\hat{x}^*$ the *minimax* value.

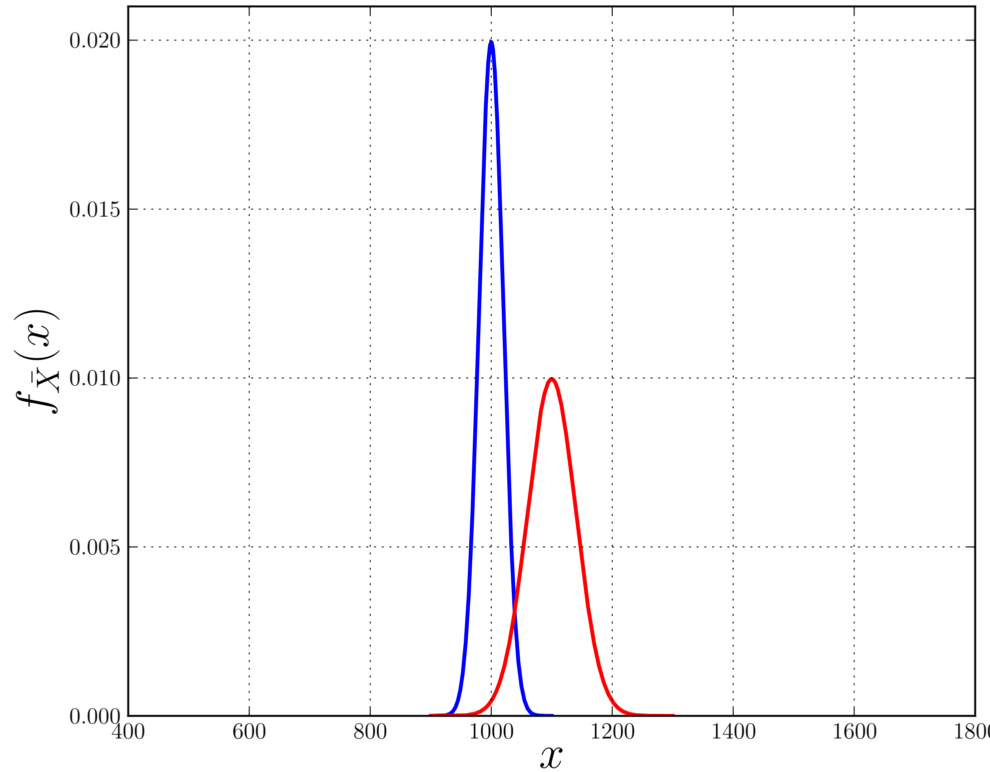
- The value of $\hat{x}^*$ depends on σ_1 and σ_2 .

- The value of $\hat{x}^*$ is independent of the sample size.

- (We will prove this!)



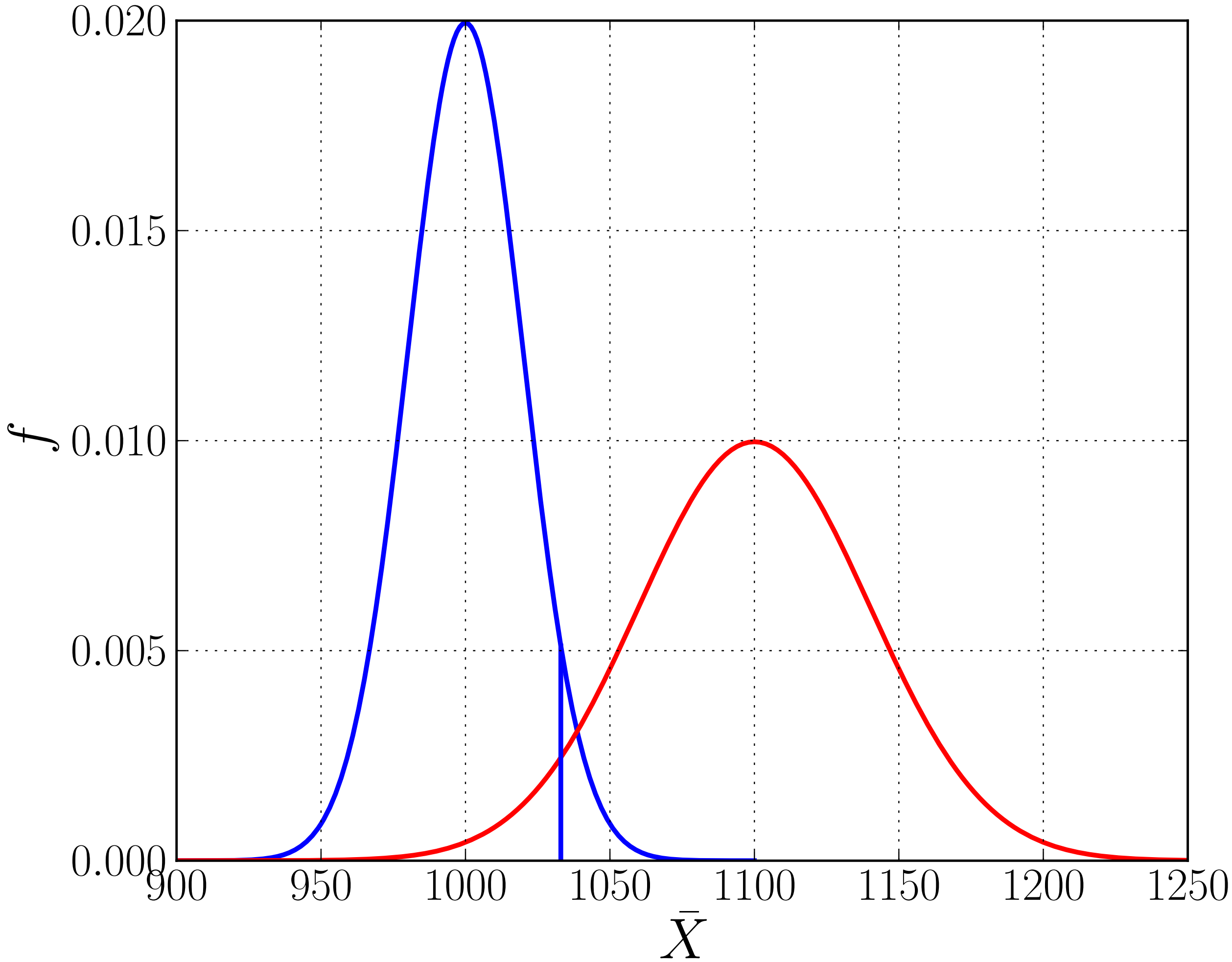
The population density functions.



The density functions of $\bar{X}$ $(n = 25)$.

$$(\mu_1 \ , \ \sigma_1) \quad = \quad (1000 \ , \ 100) \qquad (\textit{blue})$$

$$(\mu_2 \ , \ \sigma_2) \quad = \quad (1100 \ , \ 200) \qquad (\textit{red})$$



The density functions of $\bar{X}$ $(n = 25)$, with minimax value of $\hat{x}$.

$$(\mu_1, \sigma_1) \ = \ (1000, 100) \ \ (\mathit{blue}) \quad , \quad (\mu_2, \sigma_2) \ = \ (1100, 200) \ \ (\mathit{red}) \ .$$

The minimax value $\hat{x}^*$ of $\hat{x}$ is *easily computed* : At $\hat{x}^*$ we have

$$
\begin{aligned}
& P(\text{ Type 1 Error }) &=& \quad P(\text{ Type 2 Error }) \,, \\
\Longleftrightarrow \quad & P(\bar{X} \geq \hat{x}^* \mid \mu = \mu_1) &=& \quad P(\bar{X} \leq \hat{x}^* \mid \mu = \mu_2) \,, \\
\Longleftrightarrow \quad & \Phi\Big(\frac{\mu_1 - \hat{x}^*}{\sigma_1/\sqrt{n}} \Big) &=& \quad \Phi\Big(\frac{\hat{x}^* - \mu_2}{\sigma_2/\sqrt{n}} \Big) \,, \\
\Longleftrightarrow \quad & \frac{\mu_1 - \hat{x}^*}{\sigma_1/\sqrt{n}} &=& \quad \frac{\hat{x}^* - \mu_2}{\sigma_2/\sqrt{n}} \,, \qquad (\text{ by } \textit{monotonicity} \text{ of } \Phi \text{ }) \,.
\end{aligned}
$$

from which

$$\hat{x}^* \quad = \quad \frac{\mu_1 \cdot \sigma_2 \ + \ \mu_2 \cdot \sigma_1}{\sigma_1 + \sigma_2} \,. \qquad\qquad (\text{ Check ! })$$

With $\mu_1 = 1000$, $\sigma_1 = 100$, $\mu_2 = 1100$, $\sigma_2 = 200$, we have

$$\hat{x}^* \quad = \quad \frac{1000 \cdot 200 \ + \ 1100 \cdot 100}{100 + 200} \quad = \quad 1033 \,.$$

Thus we have proved the following :

FACT : Suppose Lot 1 and Lot 2 are *normally distributed* , with mean and standard deviation

$$(\mu_1 \ , \ \sigma_1) \quad \text{and} \quad (\mu_2 \ , \ \sigma_2) \ , \quad \text{where} \quad (\mu_1 < \mu_2) \ ,$$

and *sample size* n .

Then the value of *decision criterion* $\hat{x}$ that *minimizes*

$$\max \ \{ \ P(\text{Type 1 Error}) \ , \ P(\text{Type 2 Error}) \ \} \ ,$$

i.e., the value of $\hat{x}$ that minimizes

$$\max \ \{ \ P(\bar{X} \geq \hat{x} \mid \mu = \mu_1, \ \sigma = \sigma_1) \ , \ \ P(\bar{X} \leq \hat{x} \mid \mu = \mu_2, \ \sigma = \sigma_2 \ \} \ ,$$

is given by

$$\hat{x}^* \quad = \quad \frac{\sigma_1\mu_2 \ + \ \sigma_2\mu_1}{\sigma_1 \ + \ \sigma_2} \ .$$

EXERCISE :

Determine the optimal *decision criterion* $\hat{x}^*$ that *minimizes*

$$\max \ \{ \ P(\text{Type 1 Error}) \ , \ P(\text{Type 2 Error}) \ \} \ ,$$

when

$$(\mu_1, \sigma_1) = (1000, 200) \quad , \quad (\mu_2, \sigma_2) = (1100, 300) \ .$$

For this $\hat{x}^*$ find the probability of a Type 1 and a Type 2 Error ,

when

$$n = 1 \quad , \quad n = 25 \quad , \quad n = 100 \ .$$

EXAMPLE (*Known standard deviation*) :

Given : A sample of size 9 from a *normal population* with $\sigma = 0.2$ has sample mean

$$\bar{X} = \mathbf{4.88} ,$$

Claim : The population mean is

$$\mu = \mathbf{5.00} , \qquad (\text{ the ''}\textit{null hypothesis}\text{'' } H_0)$$

We see that $\quad | \bar{X} - \mu | \quad = \quad | 4.88 - 5.00 | \quad = \quad 0.12 .$

We *reject* H_0 if $P(\ | \bar{X} - \mu | \ \geq \ 0.12 \ )$ is *rather small* , say, if

$$P(\ | \bar{X} - \mu | \ \geq \ 0.12 \ ) \ < \ 10 \ \% \qquad (\text{''}\textit{level of significance}\text{''} \ \ 10 \ \% \ )$$

Do we *accept* H_0 ?

SOLUTION (*Known standard deviation*) :

Given: $n = 9$, $\sigma = 0.2$, $\bar{X} = \mathbf{4.88}$, $\mu = \mathbf{5.0}$, $\mid \bar{X} - \mu \mid = 0.12$.

Since

$$Z \equiv \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \qquad \text{is standard normal ,}$$

the "*p-value*" (from the *standard normal Table*) is

$$P(\mid \bar{X} - \mu \mid \ \geq \ 0.12 \ ) \ = \ P\Big(\mid Z \mid \ \geq \ \frac{0.12}{0.2/\sqrt{9}} \Big)$$

$$= \ P(\mid Z \mid \ \geq \ 1.8 \ ) \ = \ 2 \ \Phi(-1.8) \ \cong \ 7.18 \ \% \ .$$

Thus we *reject* the hypothesis that $\mu = 5.00$ significance level 10 % .

NOTE : We would *accept* H_0 if the level of significance were 5 % .

(We are "more tolerant" when the level of significance is smaller.)

EXAMPLE (*Unknown standard deviation, large sample*) :

Given : A sample of size $n = 64$ from a *normal population* has

$$\text{sample mean} \quad \bar{X} = \mathbf{4.847} \ ,$$

and

$$\text{sample standard deviation} \quad \hat{S} = 0.234 \ .$$

Test the hypothesis that $\mu \leq \mathbf{4.8}$,

and *reject* it if $P(\bar{X} \geq 4.847)$ is small, say if

$$P(\bar{X} \geq 4.847) < 5 \% \ .$$

NOTE : Since the sample size $n = 64$ is large, we can assume that

$$\sigma \cong \hat{S} = 0.234 \ .$$

SOLUTION (*Unknown standard deviation, large sample*) :

Given $\bar{X} = \mathbf{4.847}$, $\mu = \mathbf{4.8}$, $n = 64$, and $\sigma \cong \hat{S} = 0.234$.

Using the standard normal approximation we have that

$$\bar{X} \geq 4.847 ,$$

if and only if

$$Z \equiv \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{4.847 - 4.8}{0.234/8} = 1.6 .$$

From the standard normal Table we have the *p-value*

$$P(Z \geq 1.6) = 1 - \Phi(1.6) = \Phi(-1.6) = 5.48 \% .$$

CONCLUSION:

We (barely) *accept* H_0 at level of significance 5 % .

(We would *reject* H_0 at level of significance 10 % .)

EXAMPLE (*Unknown standard deviation, small sample*) :

A sample of size $n = 16$ from a *normal population* has

$$\text{sample mean } \bar{X} = \mathbf{4.88} \ ,$$

and

$$\text{sample standard deviation } \hat{S} = 0.234 \ .$$

Test the *null hypothesis*

$$H_0 \ : \quad \mu \ \leq \ \mathbf{4.8} \ ,$$

and *reject* it if

$$P(\bar{X} \ \geq \ \mathbf{4.88}) \ < \ 5 \ \% \ .$$

NOTE :

If $n \leq 30$ then the approximation $\sigma \ \cong \ \hat{S}$ is not so accurate.

In this case better use the "*student t-distribution*" T_{n-1} .

The T - distribution Table

n	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = \mathbf{0.01}$	$\alpha = 0.005$
5	-1.476	-2.015	-3.365	-4.032
6	-1.440	-1.943	-3.143	-3.707
7	-1.415	-1.895	-2.998	-3.499
8	-1.397	-1.860	-2.896	-3.355
9	-1.383	-1.833	-2.821	-3.250
10	-1.372	-1.812	**-2.764**	-3.169
11	-1.363	-1.796	-2.718	-3.106
12	-1.356	-1.782	-2.681	-3.055
13	-1.350	-1.771	-2.650	-3.012
14	-1.345	-1.761	-2.624	-2.977
15	-1.341	-1.753	-2.602	-2.947

This Table shows $t_{\alpha,n}$ values such that $P(T_n \leq t_{\alpha,n}) = \alpha$.

(For example, $P(T_{\mathbf{10}} \leq \mathbf{-2.764}) = \mathbf{1}\%$)

SOLUTION (*Unknown standard deviation, small sample*) :

With $n = 16$ we have $\bar{X} \geq \mathbf{4.88}$ if and only if

$$T_{n-1} \ = \ T_{15} \ = \ \frac{\bar{X} - \mu}{\hat{S}/\sqrt{n}} \ \geq \ \frac{4.88 - 4.8}{0.234/4} \ \cong \ 1.37 \ .$$

The *t-distribution Table* shows that

$$P(\ T_{15} \geq 1.341\ ) \ = \ P(\ T_{15} \leq -1.341\ ) \ = \ 10\ \%\ ,$$

$$P(\ T_{15} \geq 1.753\ ) \ = \ P(\ T_{15} \leq -1.753\ ) \ = \ 5\ \%\ .$$

Thus we *reject* H_0 at *level of significance* 10 % ,

but we *accept* H_0 at *level of significance* 5 % .

(We are "more tolerant" when the level of significance is smaller.)

EXAMPLE (*Testing a hypothesis on the standard deviation*) :

A sample of 16 items from a *normal population* has sample standard deviation $\hat{S} = \mathbf{2.58}$.

Do you believe the population standard deviation satisfies $\sigma \leq \mathbf{2.0}$?

SOLUTION : We know that

$$\frac{n-1}{\sigma^2}\,\hat{S}^2 \quad \text{has the} \quad \chi^2_{n-1} \quad \text{distribution} .$$

For our data :

$$\hat{S} \;\geq\; 2.58 \qquad \text{if and only if} \qquad \frac{n-1}{\sigma^2}\,\hat{S}^2 \;\geq\; \frac{15}{4}\,2.58^2 \;=\; 24.96 \;,$$

and from the χ^2 *Table*

$$P(\;\chi^2_{15} \;\geq\; 25.0\;) \quad \cong \quad 5.0\;\% \;.$$

Thus we (barely) *accept* the hypothesis at significance level 5 % .

(We would *reject* the hypothesis at significance level 10 % .)

EXERCISE :

A sample of 16 items from a *normal population* has sample standard deviation

$$\hat{S} = 0.83 .$$

Do you believe the *hypothesis* that σ satisfies

$$\sigma \leq 1.2 ?$$

(Probably Yes !)

Linear Least Squares

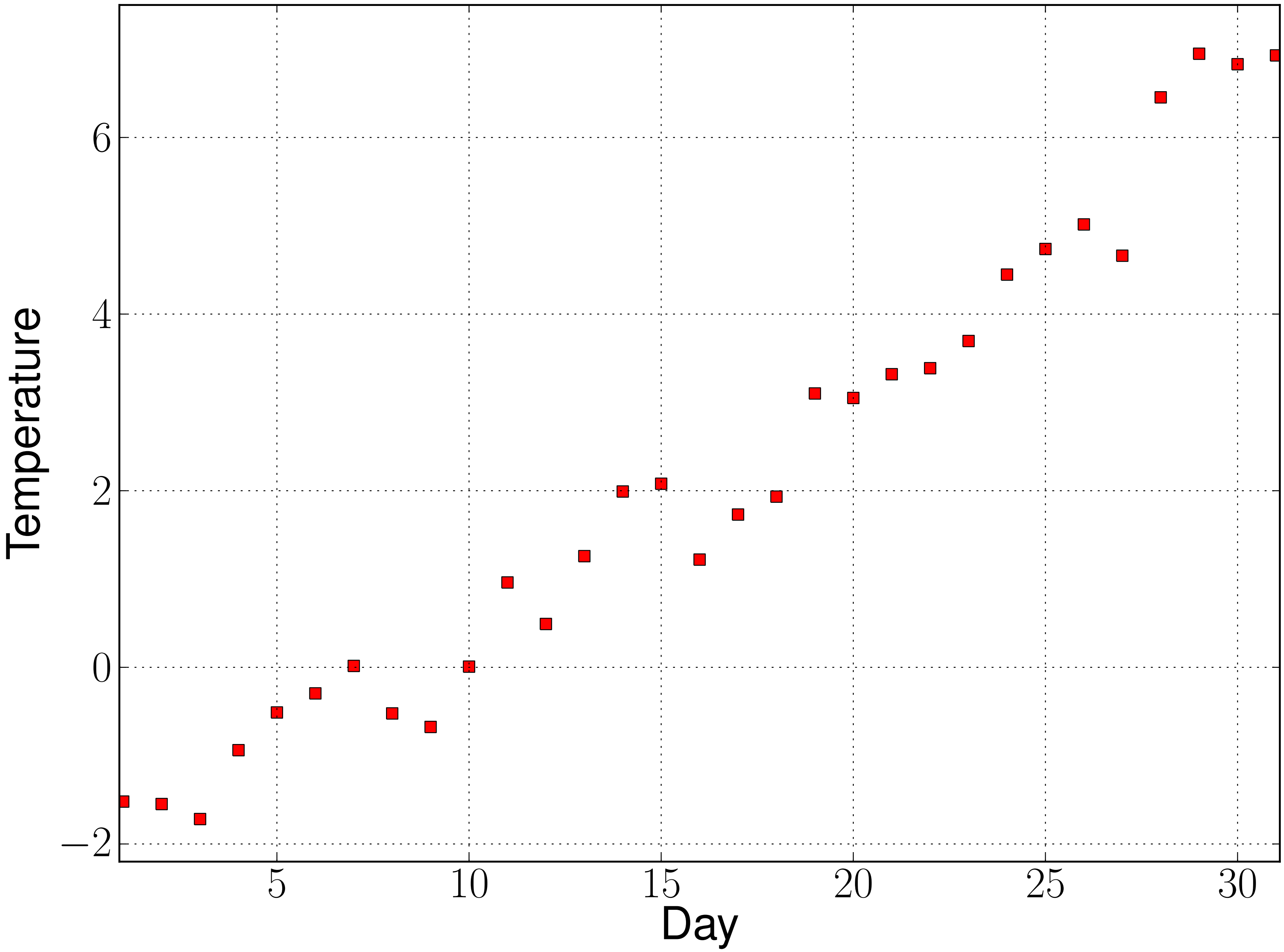
Recall the following data :

1	-1.52	8	-0.52	15	2.08	22	3.39	29	6.95
2	-1.55	9	-0.67	16	1.22	23	3.69	30	6.83
3	-1.72	10	0.01	17	1.73	24	4.45	31	6.93
4	-0.94	11	0.96	18	1.93	25	4.74		
5	-0.51	12	0.49	19	3.10	26	5.01		
6	-0.29	13	1.26	20	3.05	27	4.66		
7	0.02	14	1.99	21	3.32	28	6.45		

Average daily high temperature in Montreal in March : 1943-2014 .

(Source : `http://climate.weather.gc.ca/`)

These data have *sample correlation coefficient* $R_{X,Y} = \mathbf{0.98}$.



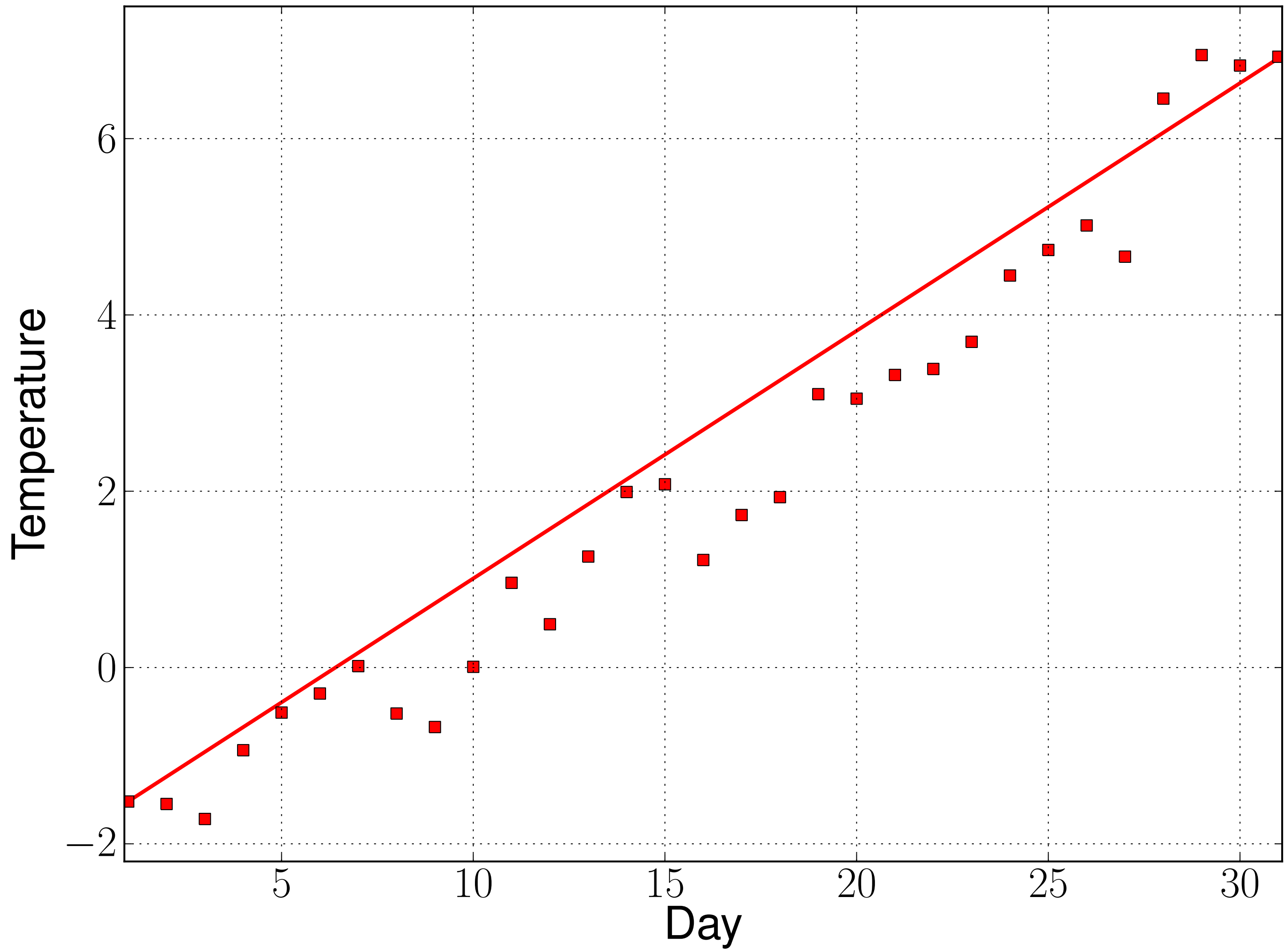
Average daily high temperature in Montreal in March

Suppose that :

- We believe that these temperatures basically increase *linearly* .
- In fact we found the *sample correlation coefficient* $R_{xy} = 0.98$.
- Thus we believe in a relation

$$T_k \quad = \quad c_1 \quad + \quad c_2 \, k \ , \qquad k \ = \ 1, \ 2, \ \cdots \ , 31 \ .$$

- The *deviations* from linearity come from *random influences* .
- These random influences can be due to *many factors* .
- The deviations may have a *normal distribution* .
- We want to determine "*the best*" linear approximation.



Average daily high temperatures, with a *linear approximation* .

QUESTION : Guess how this linear approximation was obtained !

- There are many ways to determine such a linear approximation.

- Often used is the *least squares method* .

- This method determines the values of c_1 and c_2 that *minimize* the *least squares error* :

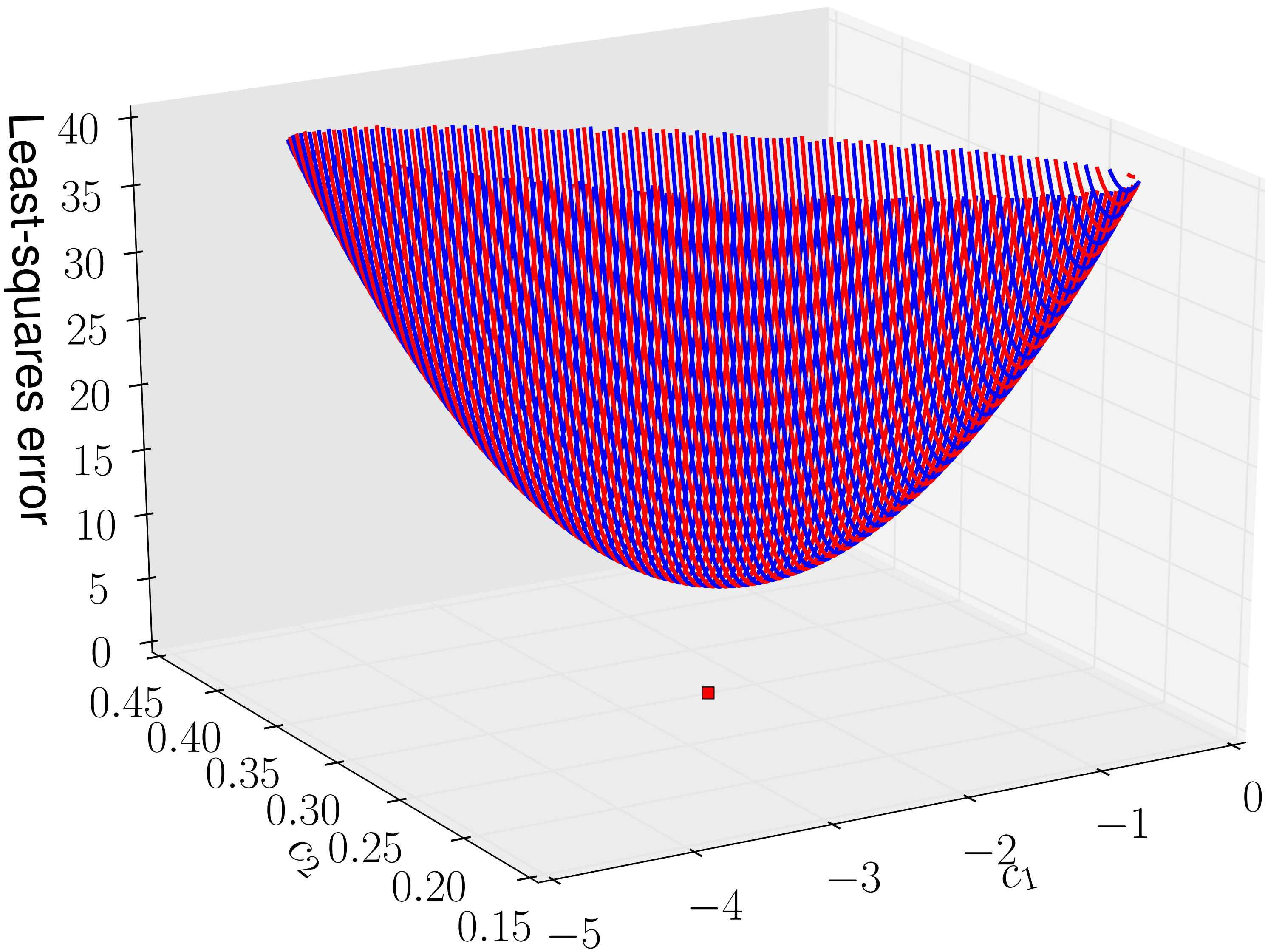
$$\sum_{k=1}^{N} (T_k - (c_1 + c_2 x_k))^2 ,$$

where, in our example, $N = 31$ and $x_k = k$.

- To do so set the *partial derivatives* w.r.t. c_1 and c_2 to zero :

$$\text{w.r.t. } c_1 : \qquad -2 \sum_{k=1}^{N} (T_k - (c_1 + c_2 x_k)) = 0 ,$$

$$\text{w.r.t. } c_2 : \qquad -2 \sum_{k=1}^{N} x_k (T_k - (c_1 + c_2 x_k)) = 0 .$$



The least squares error versus c_1 and c_2 .

From setting the partial derivatives to zero, we have

$$\sum_{k=1}^{N} (T_k - (c_1 + c_2 x_k)) = 0 \quad , \quad \sum_{k=1}^{N} x_k (T_k - (c_1 + c_2 x_k)) = 0 .$$

Solving these two equations for c_1 and c_2 gives

$$c_2 = \frac{\sum_{k=1}^{N} x_k T_k - \bar{x} \sum_{k=1}^{N} T_k}{\sum_{k=1}^{N} x_k^2 - N \bar{x}^2} ,$$

and

$$c_1 = \bar{T} - c_2 \, \bar{x} ,$$

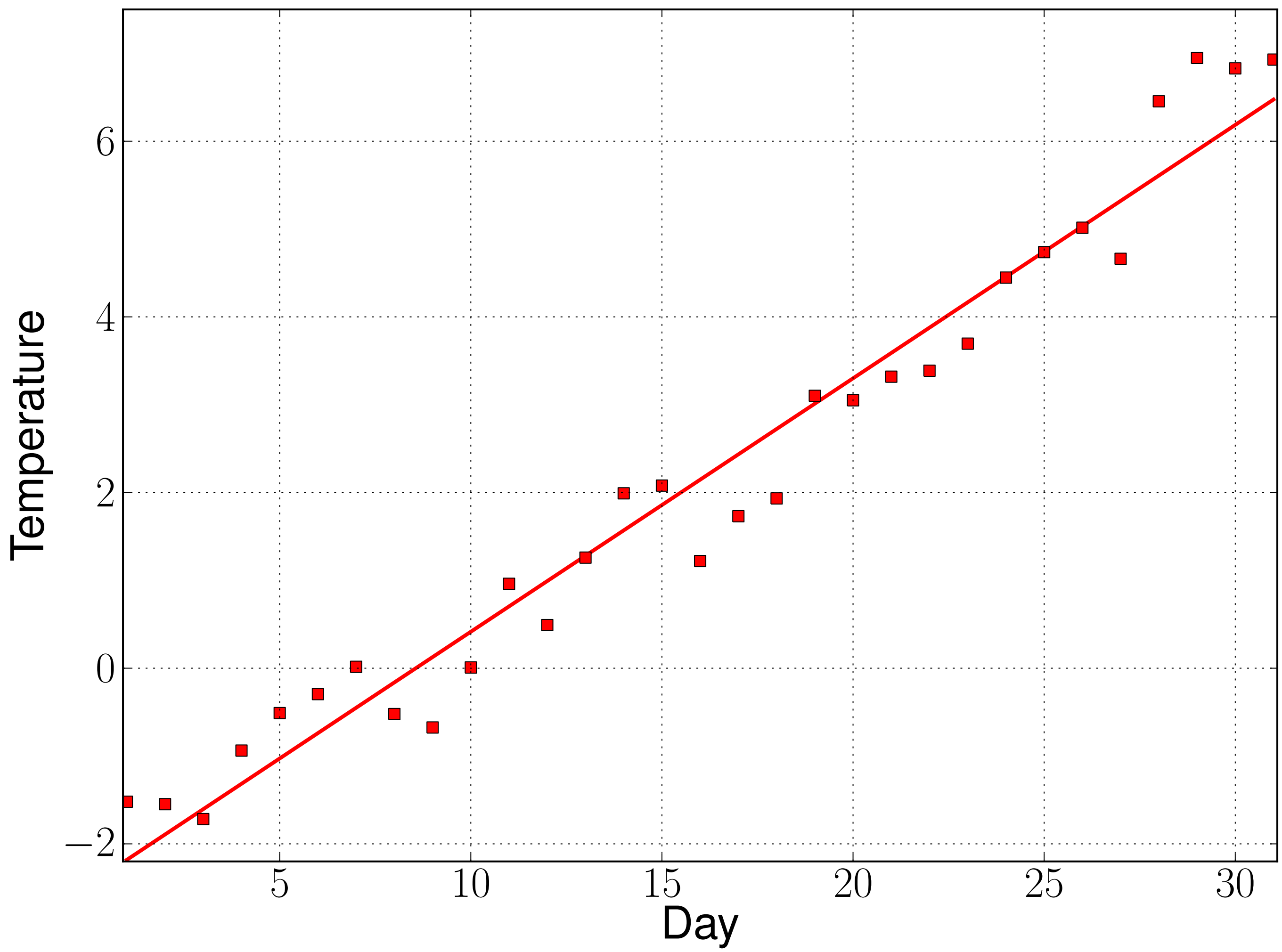
where

$$\bar{x} = \frac{1}{N} \sum_{k=1}^{N} x_k \quad , \quad \bar{T} = \frac{1}{N} \sum_{k=1}^{N} T_k .$$

EXERCISE : Check these formulas !

EXAMPLE : For our "*March temperatures*" example, we find

$$c_1 = -\ 2.47 \quad \text{and} \quad c_2 = 0.289 \ .$$



Average daily high temperatures, with linear *least squares approximation* .

General Least Squares

Given discrete data points

$$\{ \ (x_i, y_i) \ \}_{i=1}^{N} \ ,$$

find the coefficients c_k of the function

$$p(x) \quad \equiv \quad \sum_{k=1}^{n} c_k \ \phi_k(x) \ ,$$

that *minimize* the *least squares error*

$$E_L \quad \equiv \quad \sum_{i=1}^{N} \ (p(x_i) - y_i)^2$$

EXAMPLES :

- $p(x) \ = \ c_1 \ + \ c_2 \ x \ .$ (Already done !)
- $p(x) \ = \ c_1 \ + \ c_2 \ x \ + \ c_3 \ x^2 \ .$ (Quadratic approximation)

For any vector $\mathbf{x} \in \mathbb{R}^N$ let

$$\| \mathbf{x} \|^2 \quad \equiv \quad \mathbf{x}^T\mathbf{x} \quad \equiv \quad \sum_{i=1}^{N} x_k^2 \, . \qquad (T \text{ denotes } \textit{transpose}).$$

Then

$$E_L \equiv \sum_{i=1}^{N} [\, p(x_i) - y_i \,]^2 = \| \begin{pmatrix} p(x_1) \\ \cdot \\ \cdot \\ p(x_N) \end{pmatrix} - \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_N \end{pmatrix} \|^2$$

$$= \| \begin{pmatrix} \sum_{i=1}^{n} c_i\phi_i(x_1) \\ \cdot \\ \cdot \\ \sum_{i=1}^{n} c_i\phi_i(x_N) \end{pmatrix} - \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_N \end{pmatrix} \|^2$$

$$= \| \begin{pmatrix} \phi_1(x_1) & \cdot & \phi_n(x_1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi_1(x_N) & \cdot & \phi_n(x_N) \end{pmatrix} \begin{pmatrix} c_1 \\ \cdot \\ c_n \end{pmatrix} - \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_N \end{pmatrix} \|^2 \quad \equiv \quad \| \mathbf{Ac} - \mathbf{y} \|^2 \, .$$

THEOREM :

For the least squares error E_L to be *minimized* we must have

$$\mathbf{A}^T \mathbf{A} \mathbf{c} = \mathbf{A}^T \mathbf{y} .$$

PROOF :

$$\begin{aligned} E_L &= \| \mathbf{Ac} - \mathbf{y} \|^2 \\ &= (\mathbf{Ac} - \mathbf{y})^T (\mathbf{Ac} - \mathbf{y}) \\ &= (\mathbf{Ac})^T \mathbf{Ac} - (\mathbf{Ac})^T \mathbf{y} - \mathbf{y}^T \mathbf{Ac} + \mathbf{y}^T \mathbf{y} \\ &= \mathbf{c}^T \mathbf{A}^T \mathbf{Ac} - \mathbf{c}^T \mathbf{A}^T \mathbf{y} - \mathbf{y}^T \mathbf{Ac} + \mathbf{y}^T \mathbf{y} . \end{aligned}$$

PROOF : (continued $\cdots$)

We had

$$E_L = \mathbf{c}^T \mathbf{A}^T \mathbf{A} \mathbf{c} - \mathbf{c}^T \mathbf{A}^T \mathbf{y} - \mathbf{y}^T \mathbf{A} \mathbf{c} + \mathbf{y}^T \mathbf{y} \ .$$

For a *minimum* we need

$$\frac{\partial E_L}{\partial \mathbf{c}} = 0, \qquad i.e., \qquad \frac{\partial E_L}{\partial c_i} = 0 \ , \qquad i = 0, 1, \cdots, n \ ,$$

which gives

$$\mathbf{c}^T \mathbf{A}^T \mathbf{A} + (\mathbf{A}^T \mathbf{A} \mathbf{c})^T - (\mathbf{A}^T \mathbf{y})^T - \mathbf{y}^T \mathbf{A} = 0 \ , \qquad (\text{ Check ! })$$

i.e.,

$$2\mathbf{c}^T \mathbf{A}^T \mathbf{A} - 2\mathbf{y}^T \mathbf{A} = 0 \ ,$$

or

$$\mathbf{c}^T \mathbf{A}^T \mathbf{A} = \mathbf{y}^T \mathbf{A} \ .$$

Transposing gives

$$\mathbf{A}^T \mathbf{A} \mathbf{c} = \mathbf{A}^T \mathbf{y} \ . \qquad \textbf{QED !}$$

EXAMPLE : Given the data points

$$\{\ (x_i, y_i)\ \}_{i=1}^{4} \quad = \quad \{\ (0,1)\ ,\ (1,3)\ ,\ (2,2)\ ,\ (4,3)\ \}\ ,$$

find the coefficients c_1 and c_2 of $p(x) = c_1 + c_2 x$, that minimize

$$E_L \quad \equiv \quad \sum_{i=1}^{4} [\ (c_1 + c_2 x_i)\ -\ y_i]^2\ .$$

SOLUTION : Here $N = 4\ ,\ \ n = 2\ ,\ \ \phi_1(x) = 1\ ,\ \ \phi_2(x) = x$.

Use the Theorem :

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \end{pmatrix},$$

or

$$\begin{pmatrix} 4 & 7 \\ 7 & 21 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad = \quad \begin{pmatrix} 9 \\ 19 \end{pmatrix}\ ,$$

with solution $c_1 = 1.6$ and $c_2 = 0.371429$.

EXAMPLE : Given the same data points, find the coefficients of

$$p(x) = c_1 + c_2 x + c_3 x^2 ,$$

that minimize

$$E_L \equiv \sum_{i=1}^{4} [\ (c_1 + c_2\ x_i + c_3\ x_i^2) - y_i\ ]^2 .$$

SOLUTION : Here

$$N = 4 \quad , \quad n = 3 \quad , \quad \phi_1(x) = 1 \quad , \quad \phi_2(x) = x \quad , \quad \phi_3(x) = x^2 .$$

Use the Theorem :

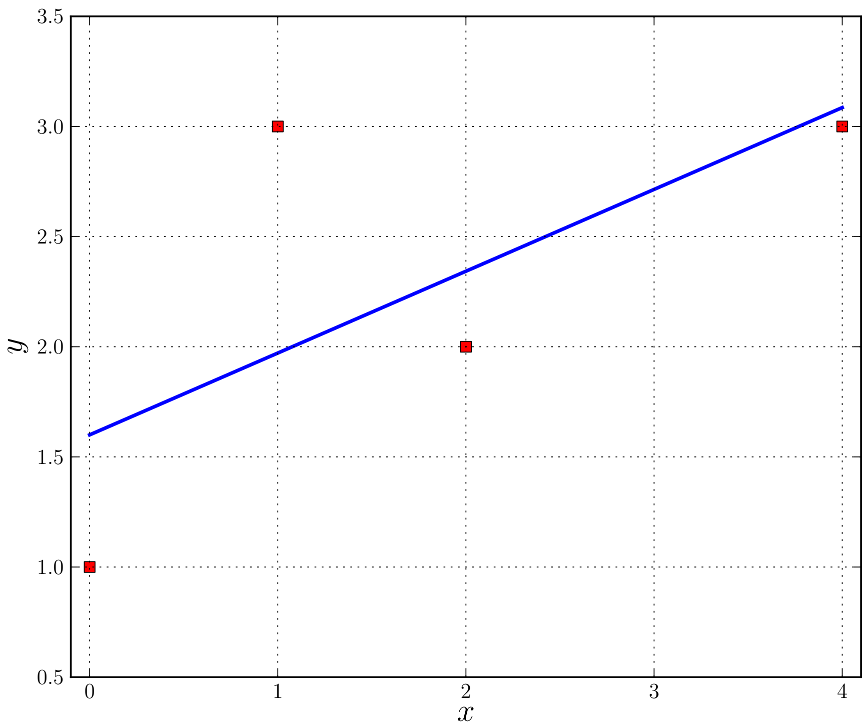
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \end{pmatrix} ,$$

or

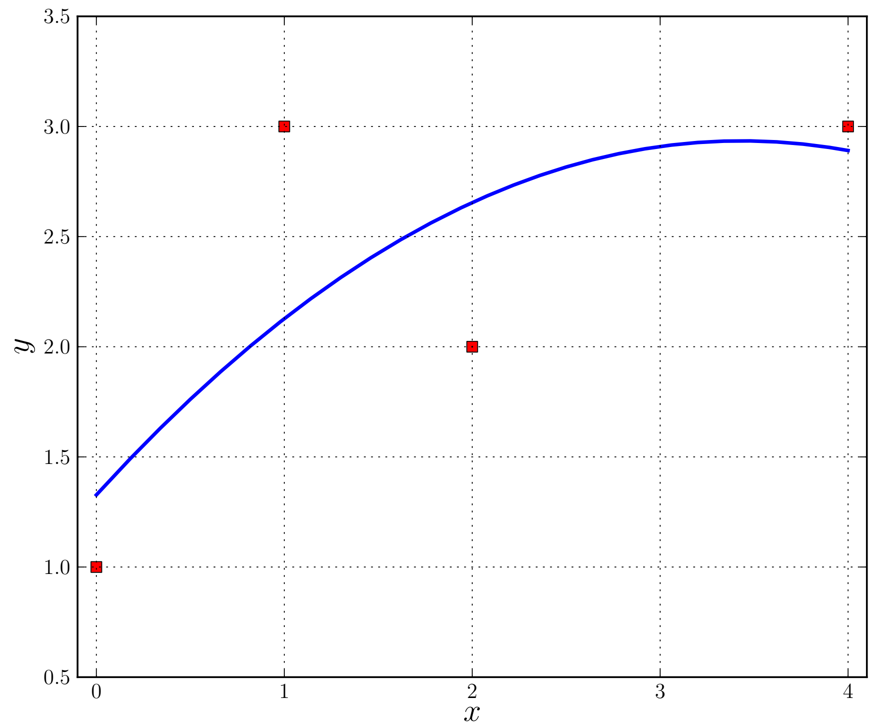
$$\begin{pmatrix} 4 & 7 & 21 \\ 7 & 21 & 73 \\ 21 & 73 & 273 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 19 \\ 59 \end{pmatrix} ,$$

with solution $c_1 = 1.32727$, $c_2 = 0.936364$, $c_3 = -0.136364$.

The least squares approximations from the preceding two examples :



$$p(x) = c_1 + c_2 x$$



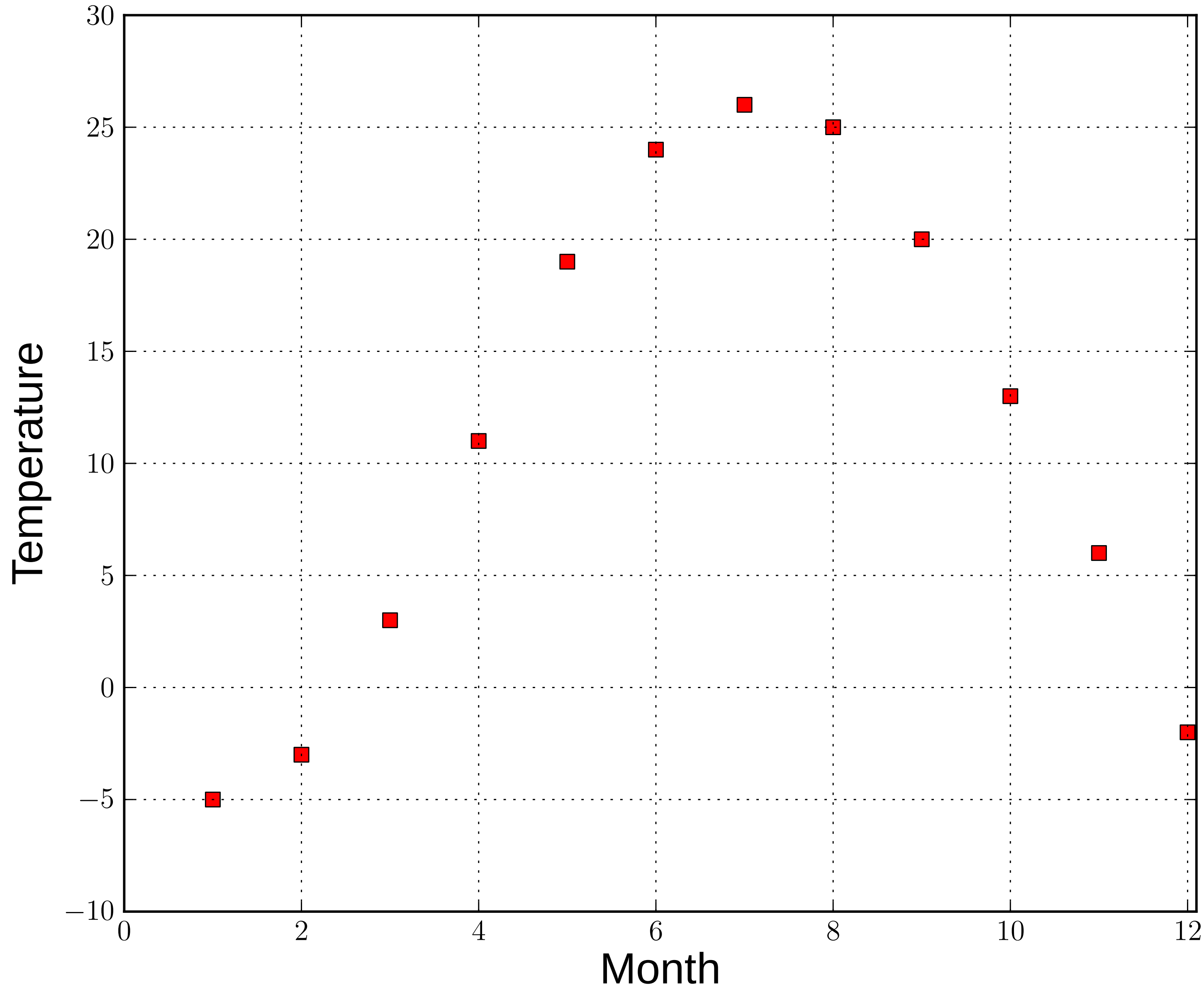
$$p(x) = c_1 + c_2 x + c_3 x^2$$

EXAMPLE : From actual data :

The average daily high temperatures in Montreal (by month) are :

January	-5
February	-3
March	3
April	11
May	19
June	24
July	26
August	25
September	20
October	13
November	6
December	-2

Source : `http : //weather.uk.msn.com`



Average daily high temperature in Montreal (by month).

EXAMPLE : (continued $\cdots$)

The graph suggests using a 3-term *least squares approximation*

$$p(x) \quad = \quad c_1\ \phi_1(x) \quad + \quad c_2\ \phi_2(x) \quad + \quad c_3\ \phi_3(x)\ ,$$

of the form

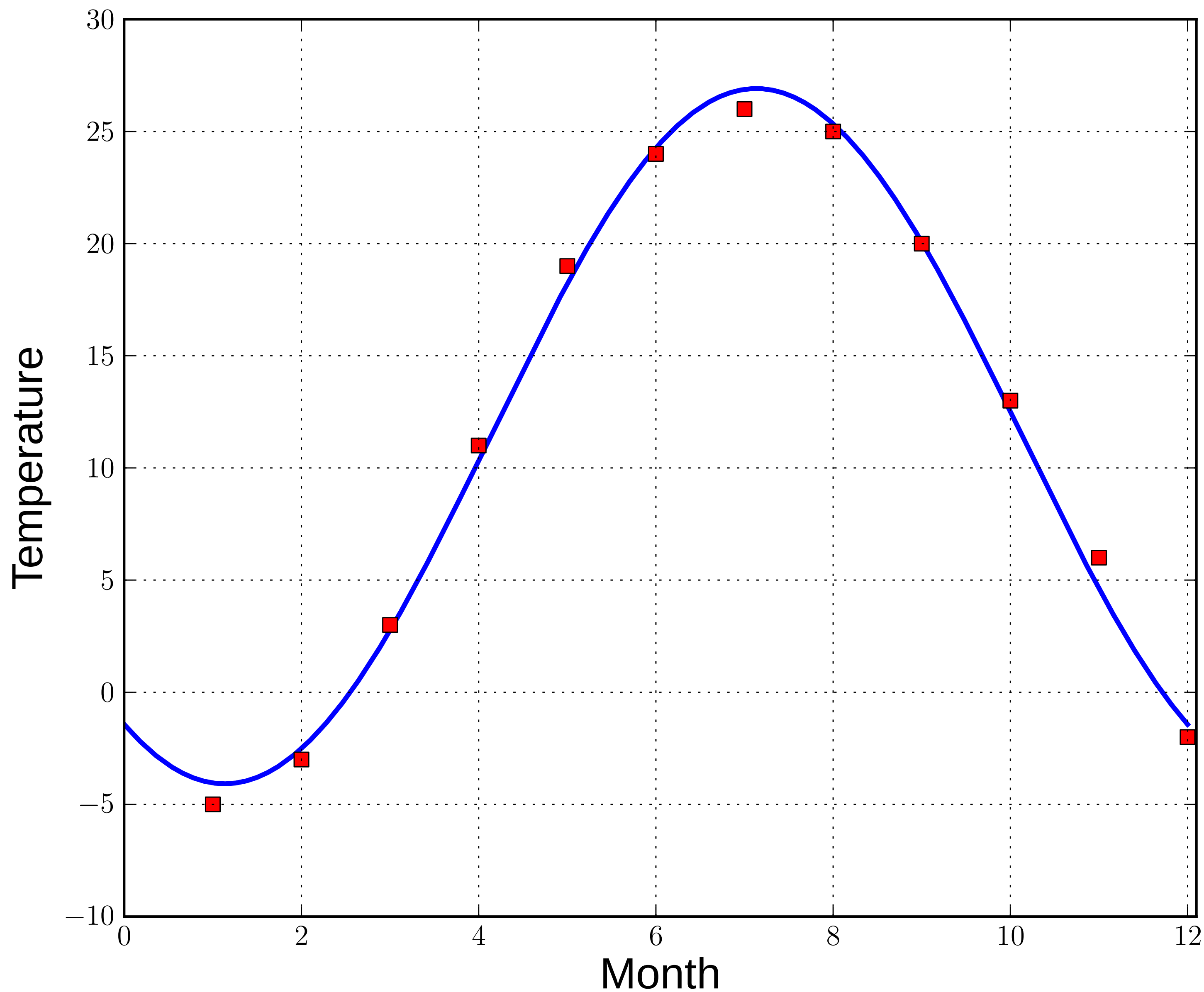
$$p(x) \quad = \quad c_1 \quad + \quad c_2\ \sin(\frac{\pi x}{6}) \quad + \quad c_3\ \cos(\frac{\pi x}{6})\ .$$

QUESTIONS :

- Why include $\phi_2(x) \ = \ \sin(\frac{\pi x}{6})$?
- Why is the argument $\frac{\pi x}{6}$?
- Why include the constant term $\phi_1(x) \ = \ c_1$?
- Why include $\phi_3(x) \ = \ \cos(\frac{\pi x}{6})$?

In this example we find the least squares coefficients

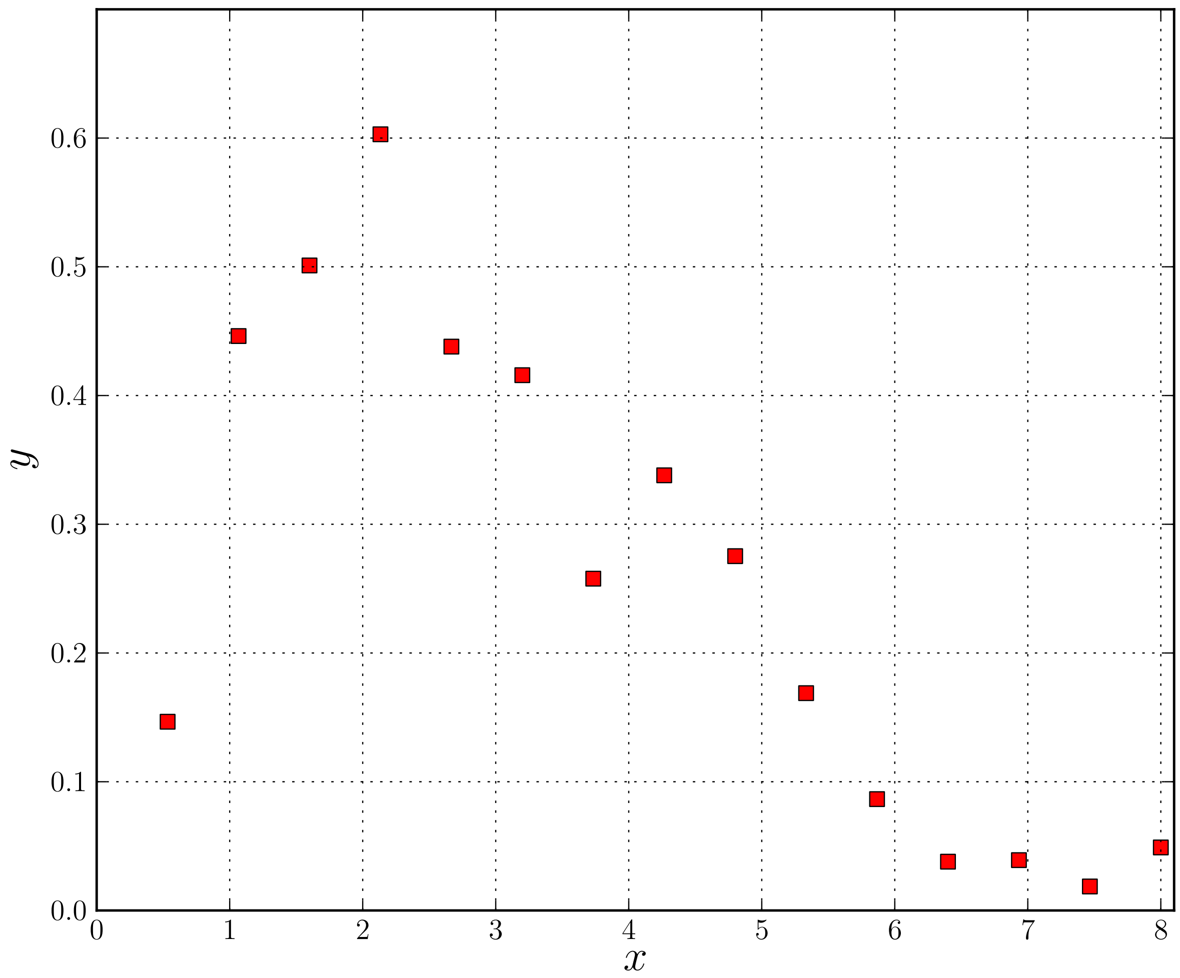
$$c_1 = 11.4 \quad , \quad c_2 = -8.66 \quad , \quad c_3 = -12.8\ .$$



Least squares fit of average daily high temperatures.

EXAMPLE :

Consider the following *experimental data* :



EXAMPLE : (continued $\cdots$)

Suppose we are given that :

- These data contain "*noise*" .
- The underlying physical process is understood.
- The *functional dependence* is *known* to have the form

$$y \quad = \quad c_1 \, x^{c_2} \, e^{-c_3 x} \ .$$

- The values of c_1 , c_2 , c_3 are *not* known.

EXAMPLE : (continued $\cdots$)

The functional relationship has the form

$$y \quad = \quad c_1 \, x^{c_2} \, e^{-c_3 x} \; .$$

NOTE :

- The unknown coefficients c_1 , c_2 , c_3 appear *nonlinearly* !
- This gives *nonlinear equations* for c_1 , c_2 , c_3 !
- Such problems are more *difficult* to solve !
- What to do ?

EXAMPLE : (continued $\cdots$)

Fortunately, in this example we can take the *logarithm* :

$$\log y \quad = \quad \log\,(\ c_1\ x^{c_2}\ e^{-c_3 x}\ ) \quad = \quad \log c_1 \ + \ c_2 \log x \ - \ c_3\, x \ .$$

This gives a *linear* relationship

$$\log y \quad = \quad \hat{c}_1\ \phi_1(x) \ + \ c_2\ \phi_2(x) \ + \ c_3\ \phi_3(x)\ ,$$

where

$$\hat{c}_1 \quad = \quad \log c_1\ .$$

and

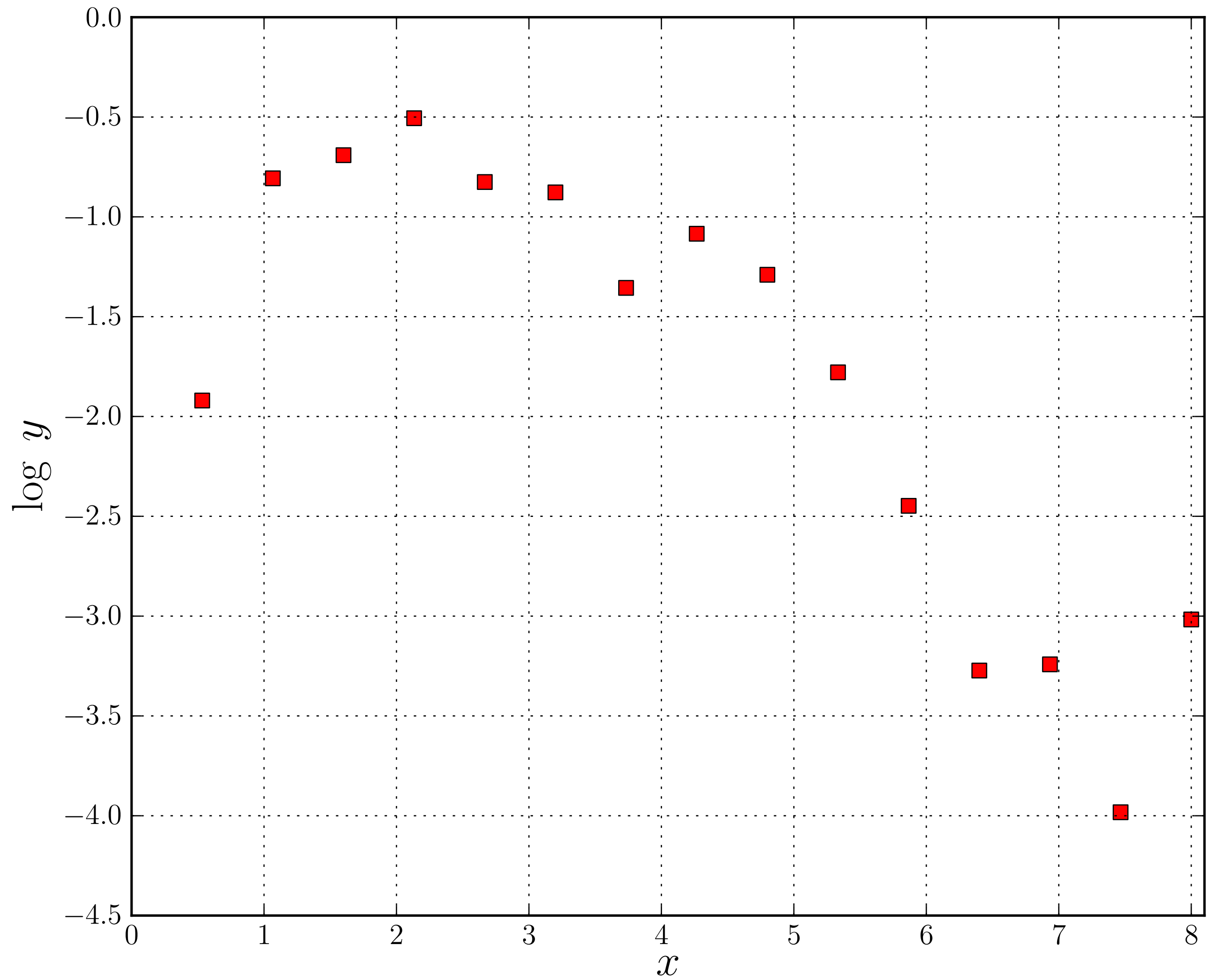
$$\phi_1(x) \ = \ 1 \quad , \quad \phi_2(x) \ = \ \log x \quad , \quad \phi_3(x) \ = \ -\,x\ .$$

Thus

- We can now use regular least squares.

- We first need to take the logarithm of the data.

EXAMPLE : (continued $\cdots$)



The logarithm of the original y-values versus x .

EXAMPLE : (continued $\cdots$)

We had

$$y \quad = \quad c_1 \ x^{c_2} \ e^{-c_3 x} \ ,$$

and

$$\log y \quad = \quad \hat{c}_1 \ \phi_1(x) \quad + \quad c_2 \ \phi_2(x) \quad + \quad c_3 \ \phi_3(x) \ ,$$

with

$$\phi_1(x) \ = \ 1 \quad , \quad \phi_2(x) \ = \ \log x \quad , \quad \phi_3(x) \ = \ - x \ ,$$

and

$$\hat{c}_1 \quad = \quad \log c_1 \ .$$

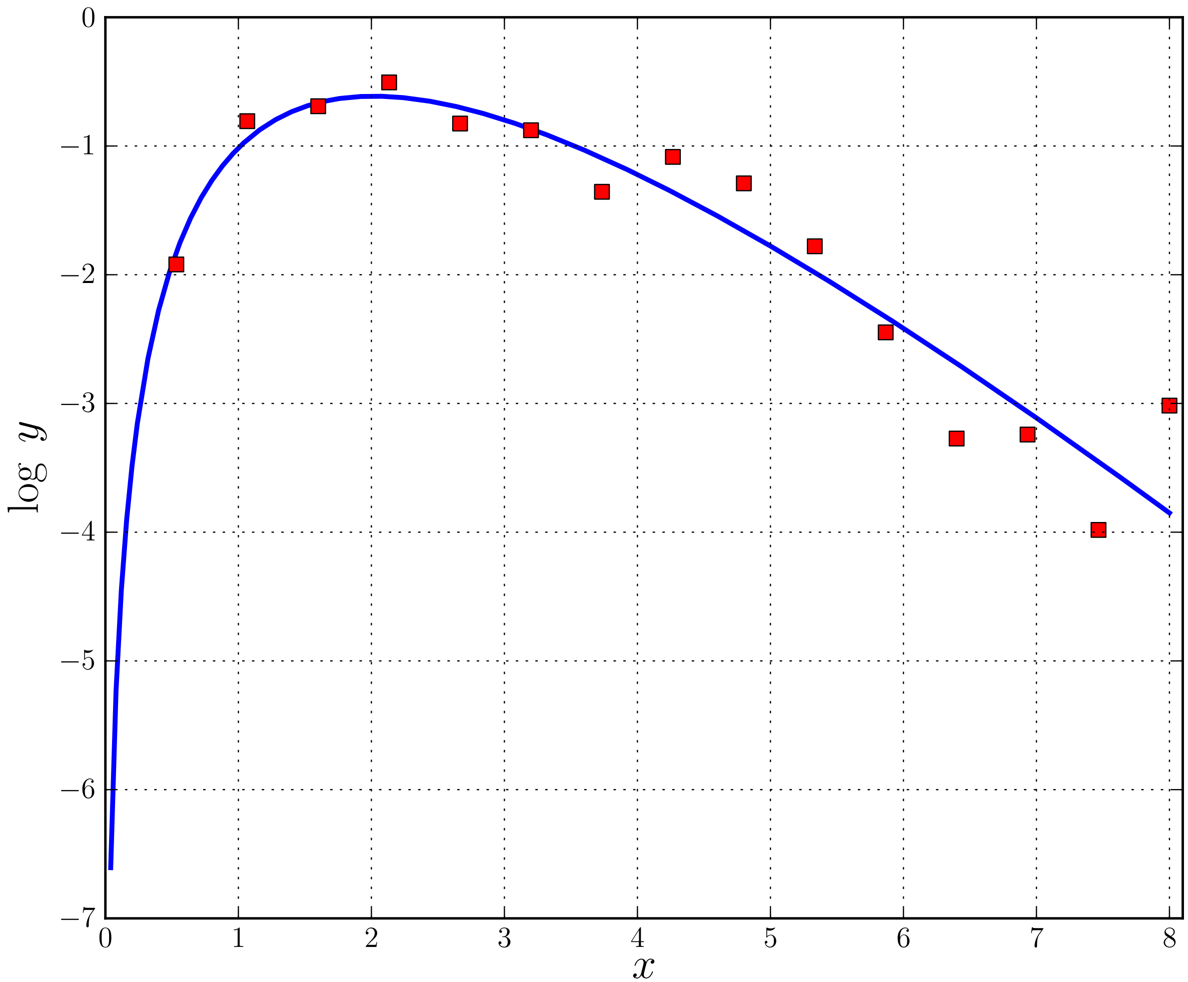
We find the following least squares values of the coefficients :

$$\hat{c}_1 \ = \ -0.00473 \quad , \quad c_2 \ = \ 2.04 \quad , \quad c_3 \ = \ 1.01 \ ,$$

and

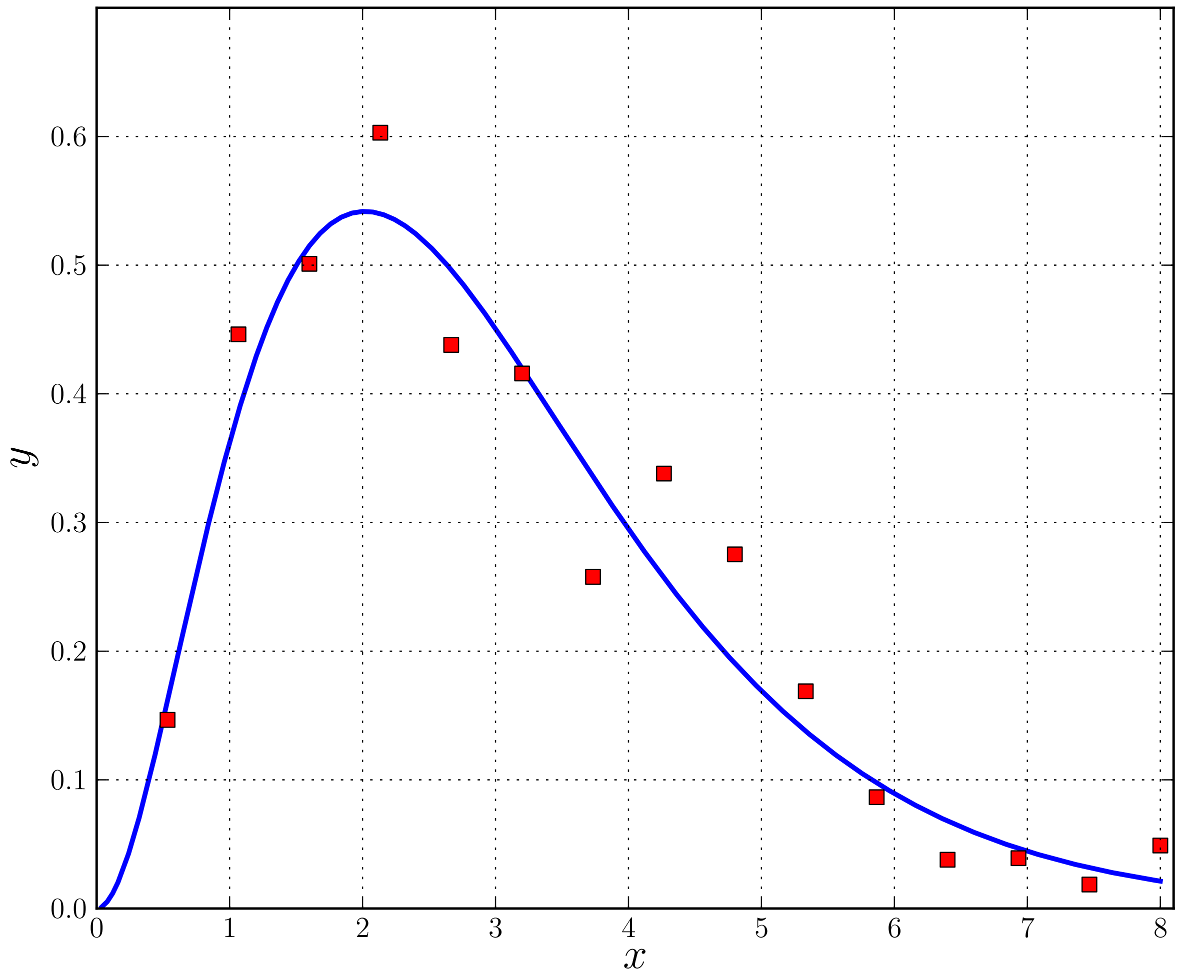
$$c_1 \quad = \quad e^{\hat{c}_1} \quad = \quad 0.995 \ .$$

EXAMPLE : (continued $\cdots$)



The least squares approximation of the transformed data.

EXAMPLE : (continued $\cdots$)



The least squares approximation shown in the original data.

RANDOM NUMBER GENERATION

- Measured data often have *random fluctuations* .
- This may be due to *inaccurate measurements* .
- It may also be due to other *external influences* .

- Often we know or believe there is a *deterministic model* .
- ($i.e.$, the process can be modeled by a deterministic *equation* .)

- However, deterministic equations can also have *random behavior* !

- The study of such equations is sometimes called *chaos theory* .
- We will look at a simple example, namely, the *logistic equation* .

The Logistic Equation

A simple deterministic model of *population growth* is

$$x_{k+1} \quad = \quad \lambda \, x_k \ , \qquad k = 1, 2, \ \cdots \ ,$$

for given λ , $(\lambda \geq 0)$, and for given x_0 , $(x_0 \geq 0)$.

The solution is

$$x_k \quad = \quad \lambda^k \ \ x_0 \ , \qquad k = 1, 2, \ \cdots \ .$$

Thus

If $0 \leq \lambda < 1$ then $x_k \rightarrow 0$ as $k \rightarrow \infty$ ("extinction") .

If $\lambda > 1$ then $x_k \rightarrow \infty$ as $k \rightarrow \infty$ ("exponential growth") .

A somewhat more realistic population growth model is

$$x_{k+1} \quad = \quad \lambda \, x_k \, (1 - x_k) \, , \qquad k = 1, 2, \quad \cdots \quad ,$$

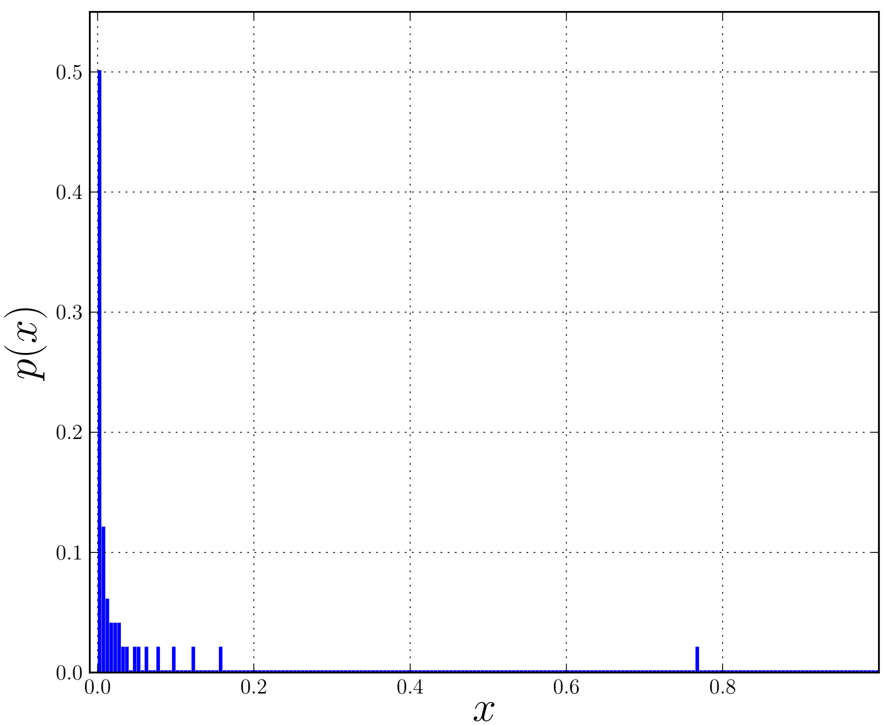
- This model is known as the *logistic equation* .
- The *maximum sustainable population* is 1 .
- λ is given, $(0 \leq \lambda \leq 4)$.
- x_0 is given, $(0 \leq x_0 \leq 1)$.
- Then $0 \leq x_k \leq 1$ for all k . (Prove this !)

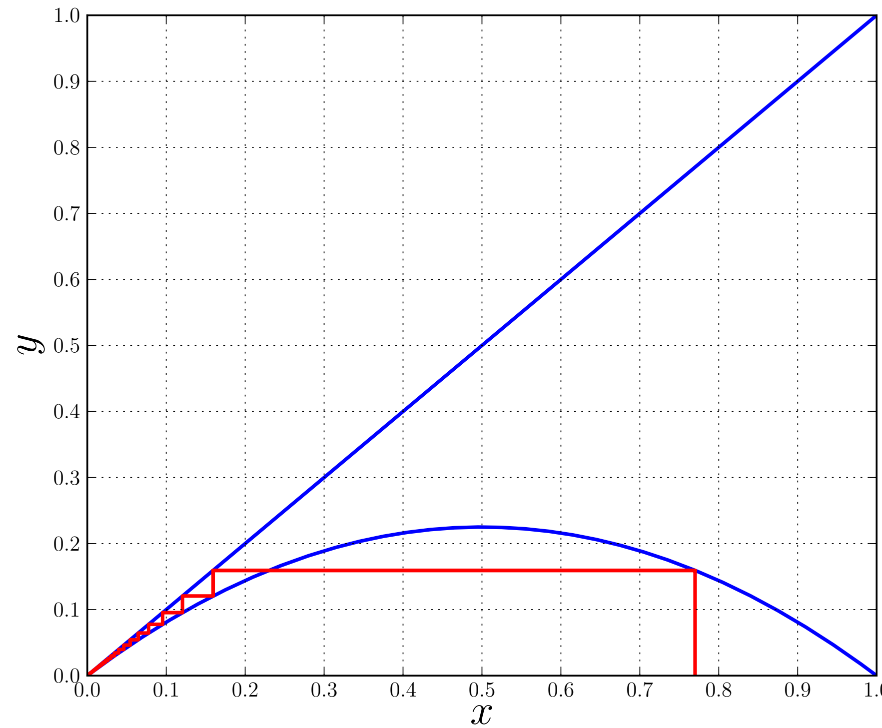
QUESTION : How does the sequence $\{x_k\}_{k=1}^{\infty}$ depend on λ ?

EXERCISE :

$$x_{k+1} \;=\; \lambda\, x_k\, (1 - x_k)\ , \qquad k = 1, 2, \ \cdots\ ,$$

- Divide the interval $[0, 1]$ into 200 *subintervals* .
- Compute x_k for $k = 1, 2, \ \cdots\ 50$.
- *Count* the x_k's in each subinterval.
- Determine the *percentage* of x_k's in each subinterval.
- Present the result in a *diagram* .
- Do this for different choices of λ $(0 \leq \lambda \leq 4)$.

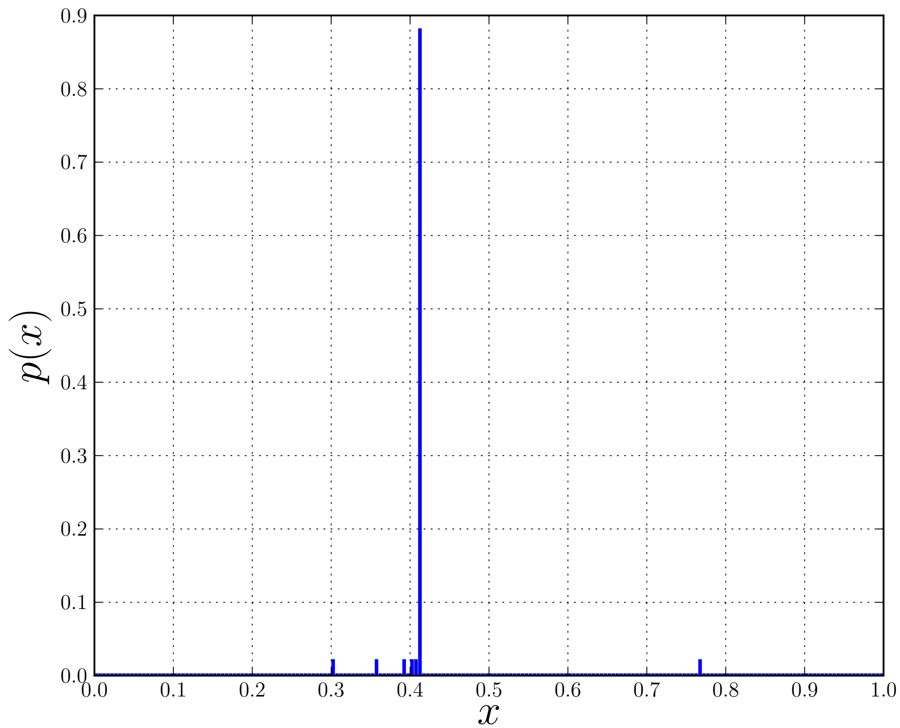


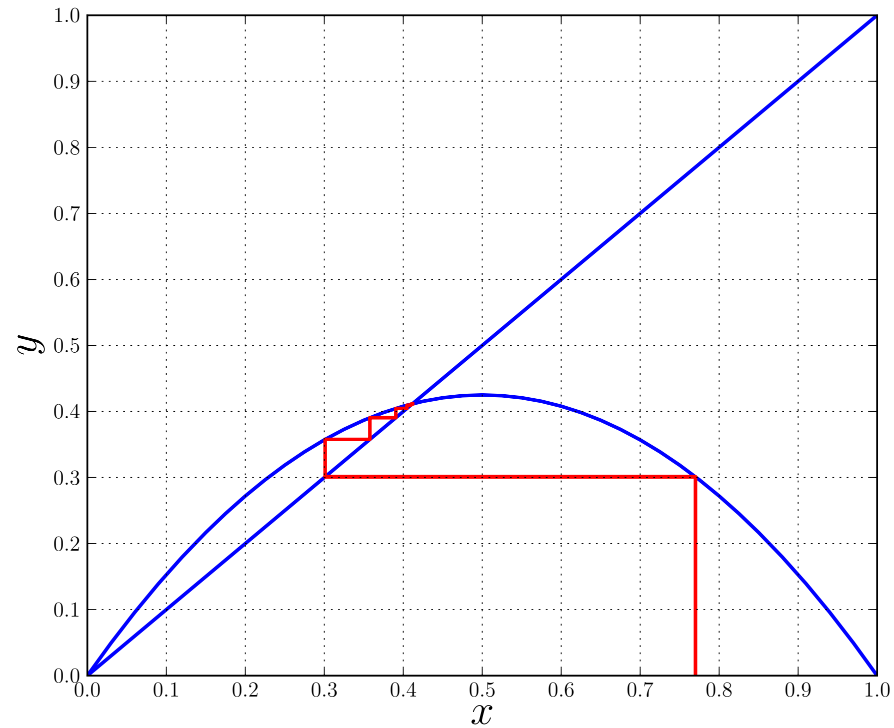


Percentage per interval

Graphical interpretation.

$\lambda = 0.9$, $x_0 = 0.77$, 50 iterations , 200 subintervals .

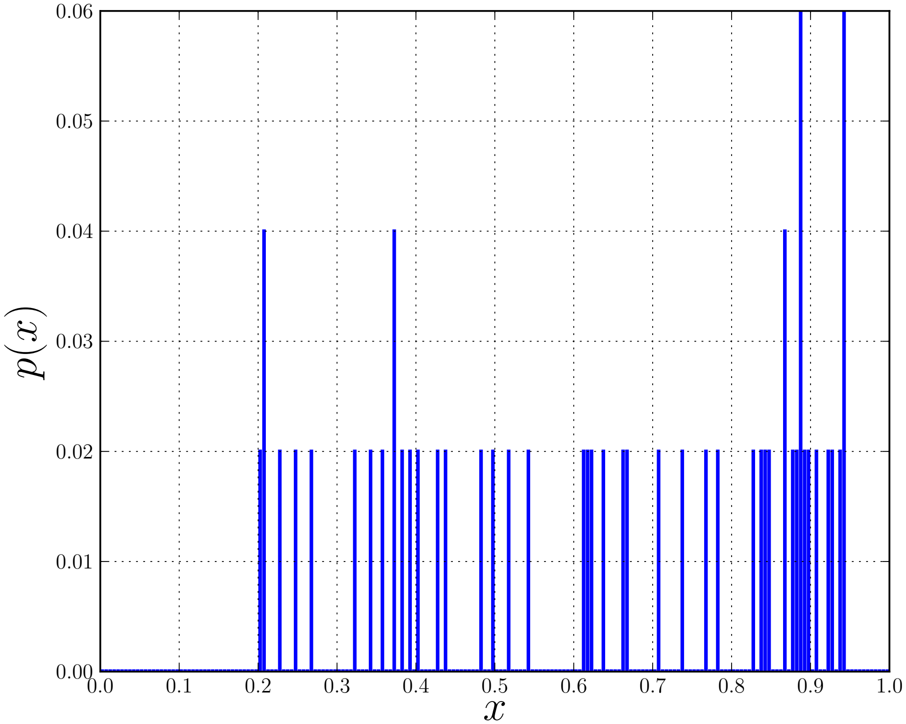


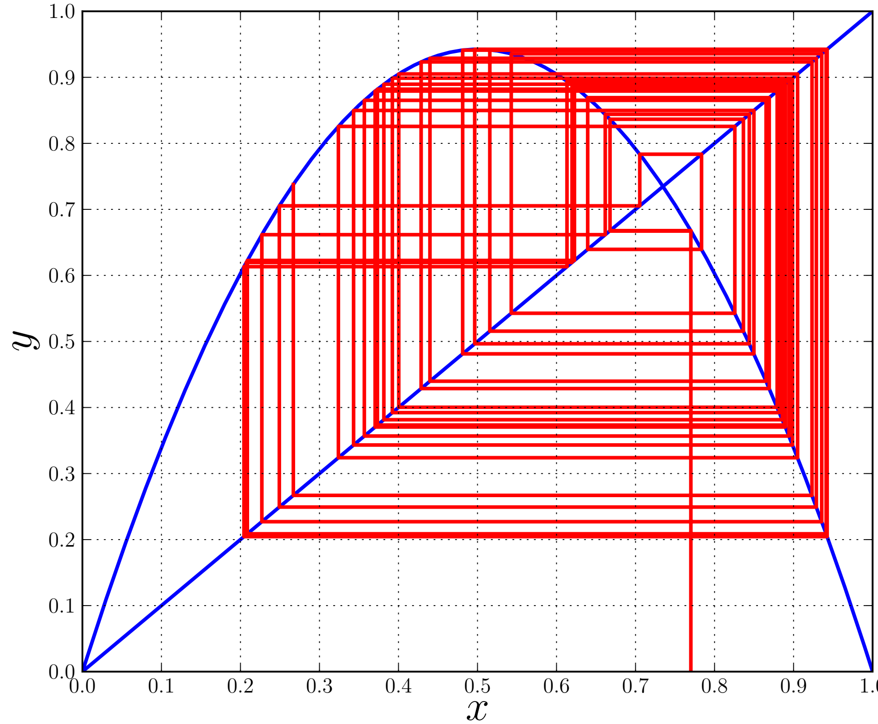


Percentage per interval

Graphical interpretation.

$\lambda = 1.7$, $x_0 = 0.77$, 50 iterations , 200 subintervals .





Percentage per interval

Graphical interpretation.

$$\lambda = 3.77 \quad , \quad x_0 = 0.77 \quad , \quad 50 \text{ iterations} \quad , \quad 200 \text{ subintervals} \; .$$

EXERCISE :

$$x_{k+1} \quad = \quad \lambda \; x_k \; (1 - x_k) \; , \qquad k = 1, 2, \; \cdots \; ,$$

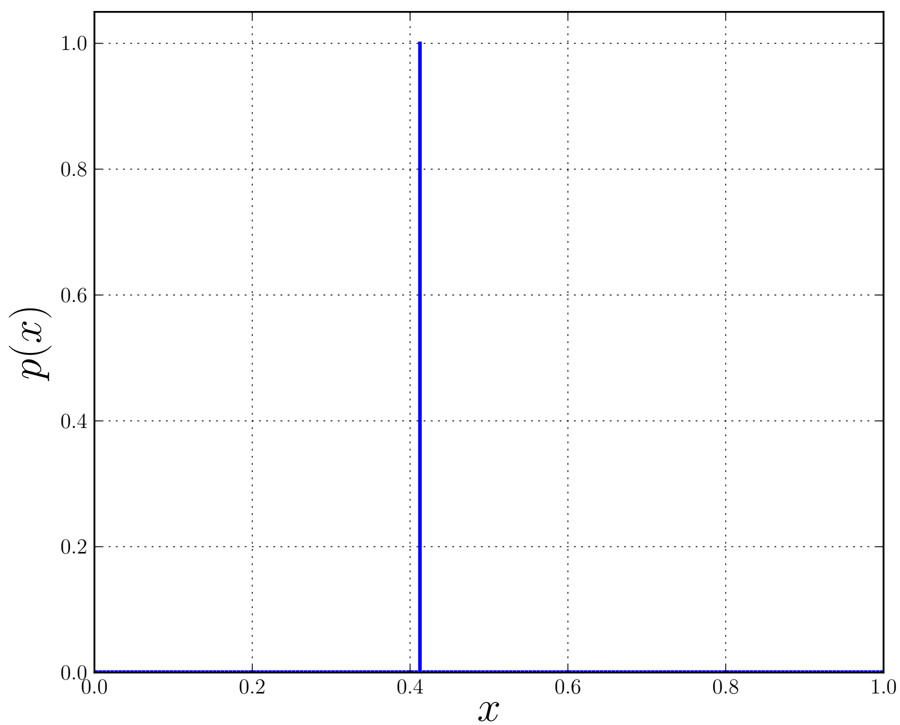
Do the same as in the preceding example, but now

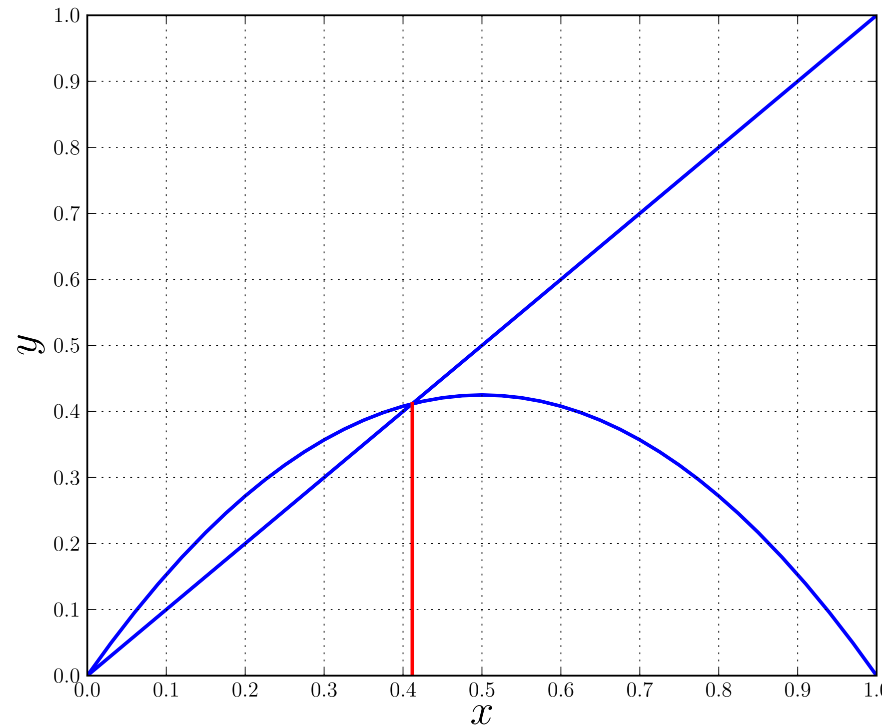
- Compute x_k for $k = 1, 2, \; \cdots \; 1,000,000$!

- Do not record the first 200 iterations.

- (This to eliminate *transient effects* .)

- You will see that there is a *fixed point* (a *cycle of period 1*) .

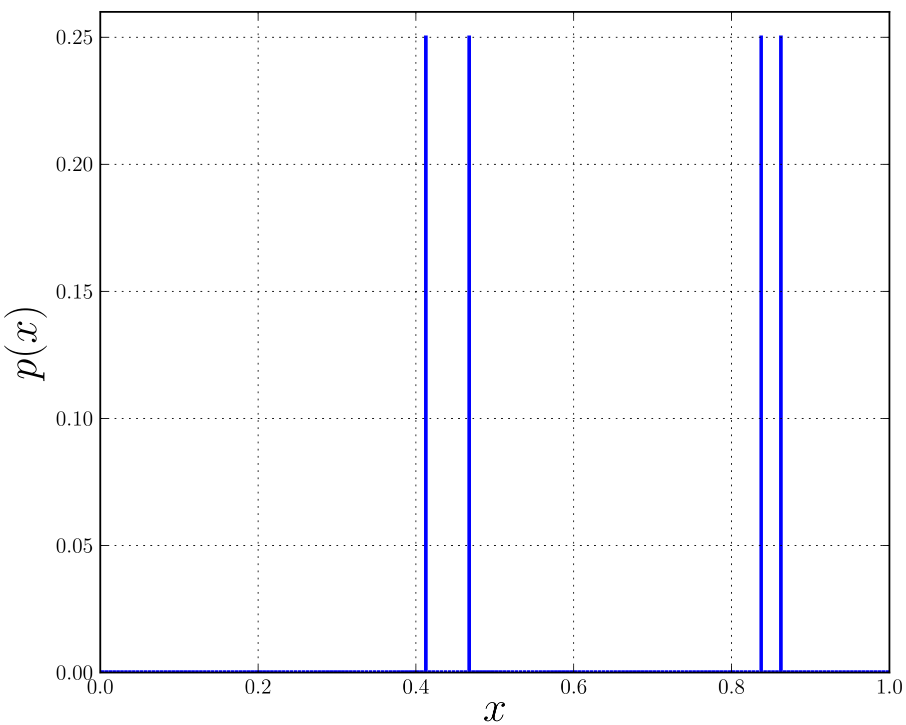


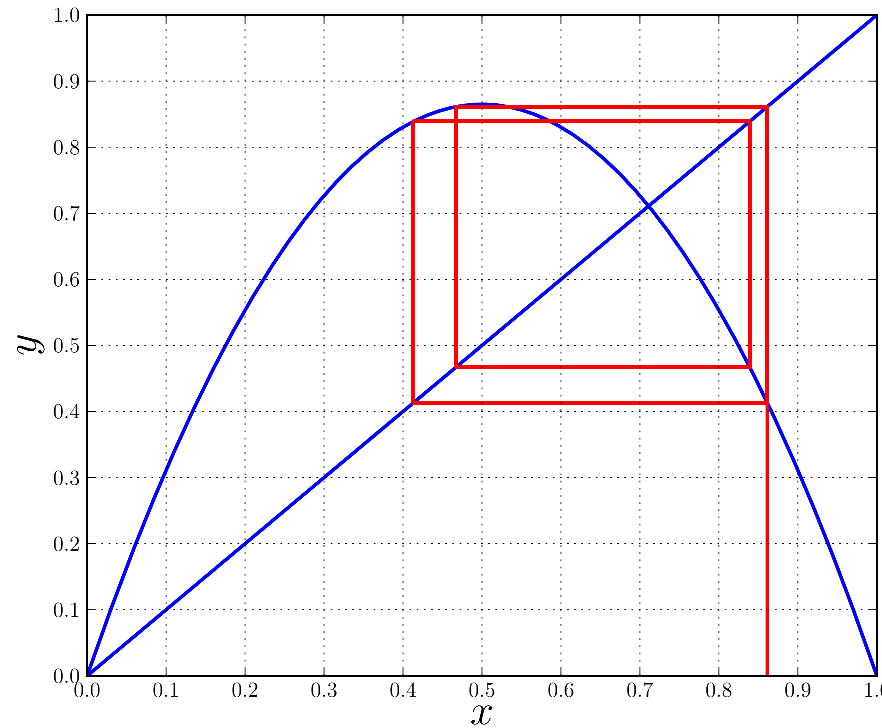


Percentage per interval — Graphical interpretation.

$\lambda = 1.7$, $1,000,000$ iterations , 200 subintervals .

There is a *fixed point* (a *cycle of period 1*) .



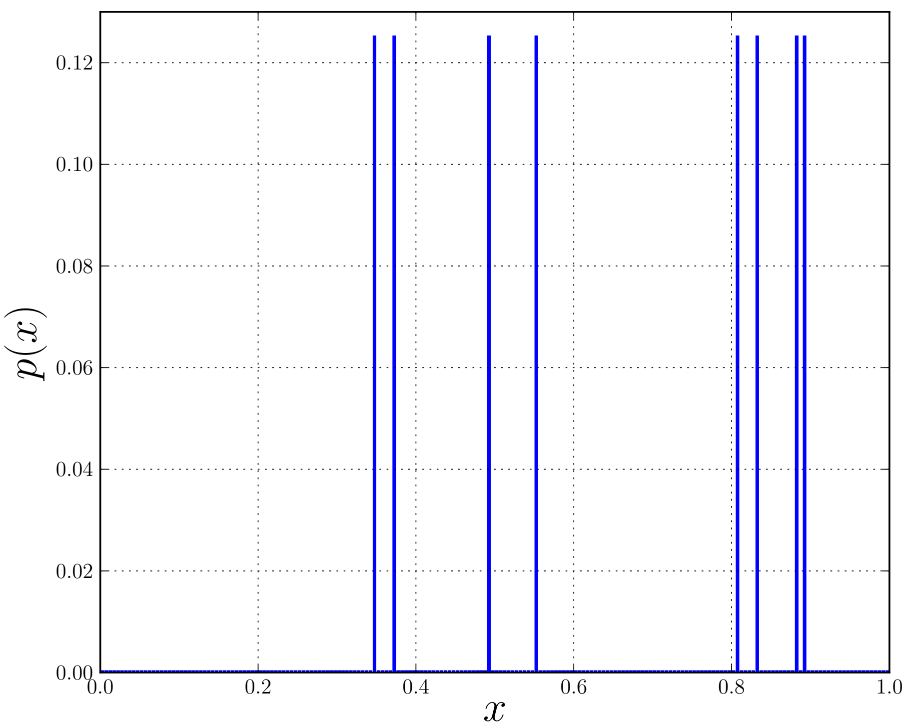


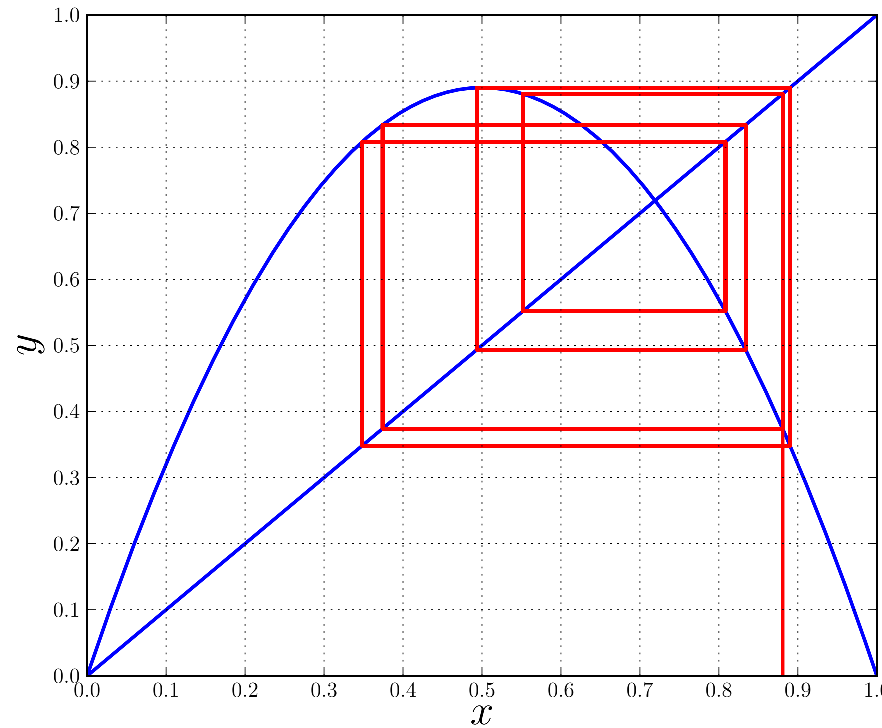
Percentage per interval

Graphical interpretation.

$\lambda = 3.46$, $1,000,000$ iterations , 200 subintervals .

There is a *cycle of period 4* .



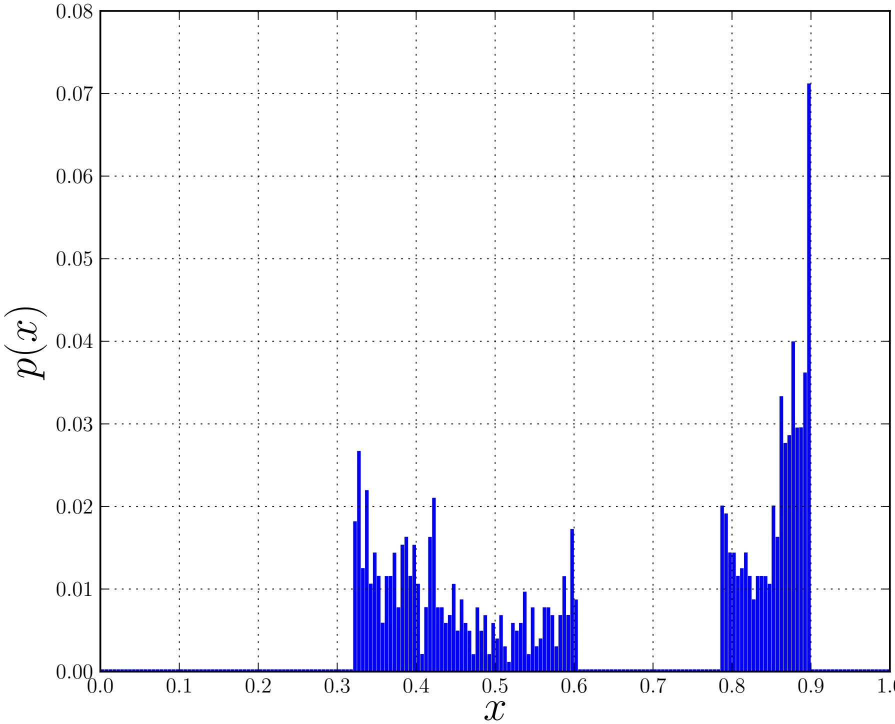


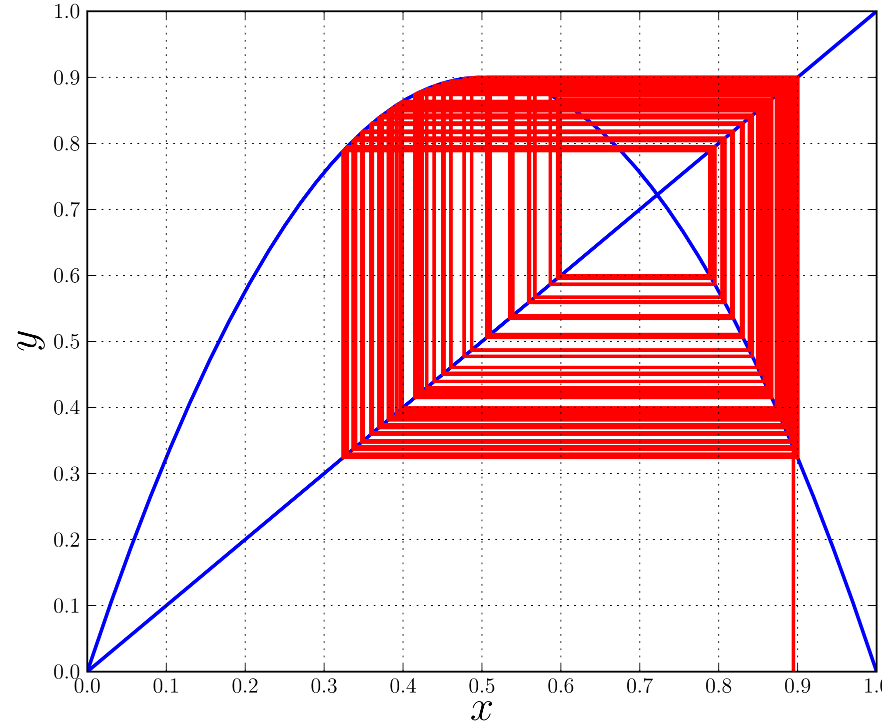
Percentage per interval

Graphical interpretation.

$\lambda = 3.561$, $1,000,000$ iterations , 200 subintervals .

There is a *cycle of period 8* .



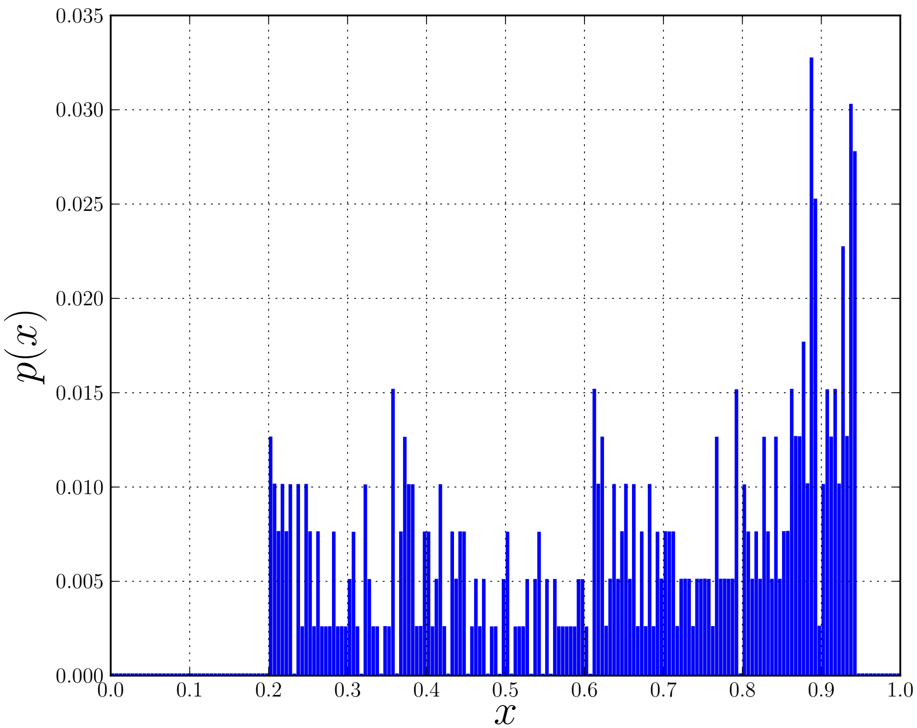


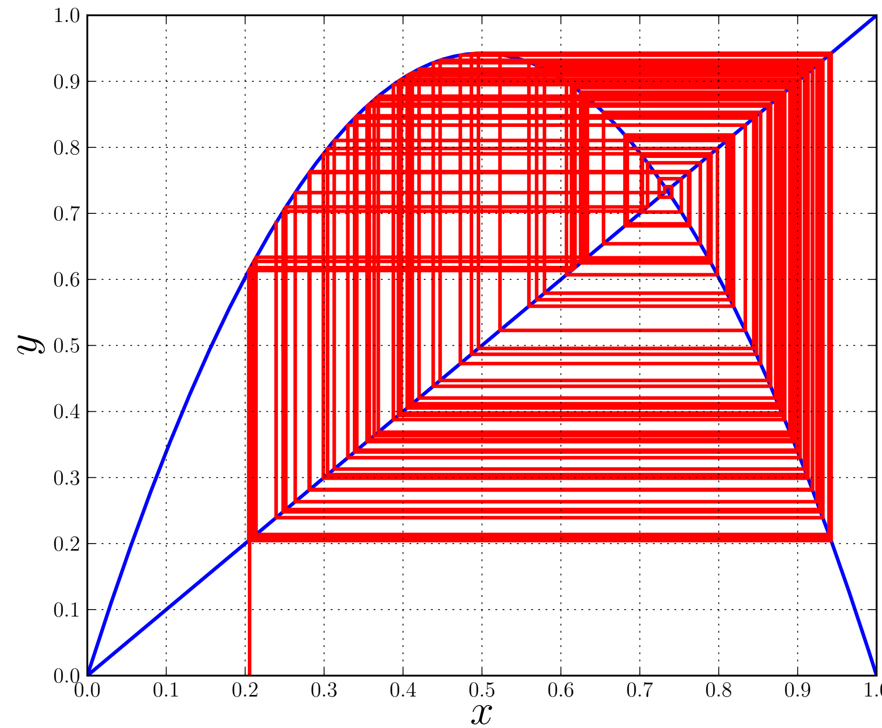
Percentage per interval Graphical interpretation.

$\lambda = 3.6$, $1,000,000$ iterations , 200 subintervals .

(The Figure on the right only shows 100 of these iterations.)

There is apparent *chaotic* behavior .



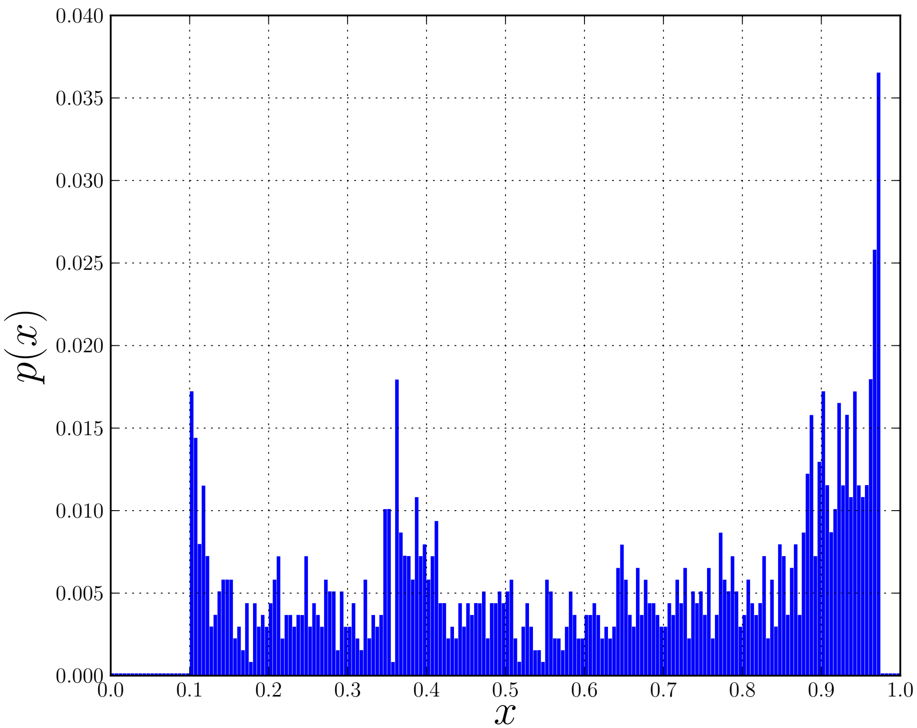


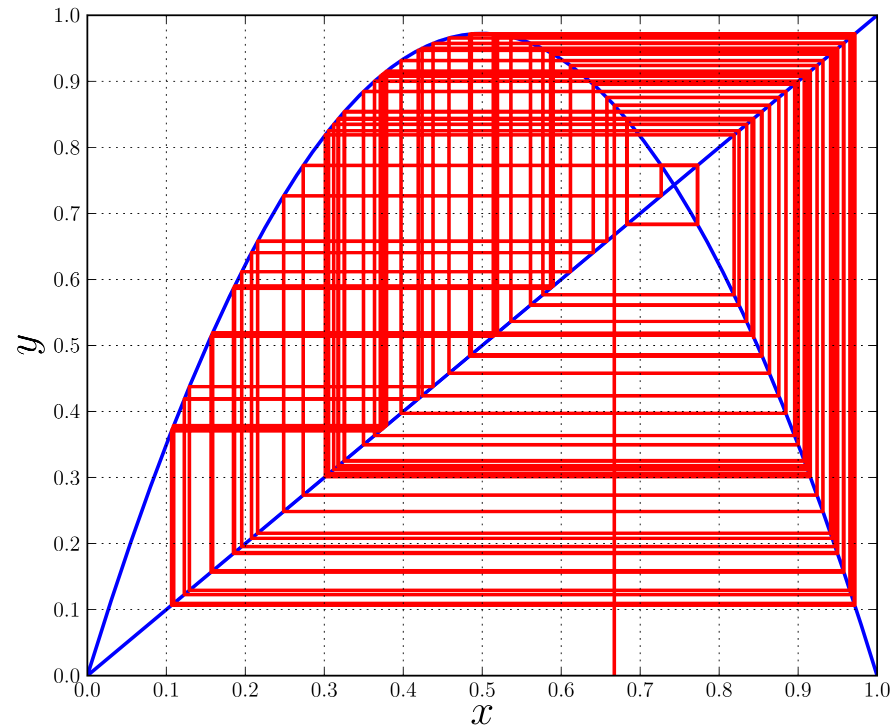
Percentage per interval Graphical interpretation.

$\lambda = 3.77$, $1,000,000$ iterations , 200 subintervals .

(The Figure on the right only shows 100 of these iterations.)

There is apparent *chaotic* behavior .



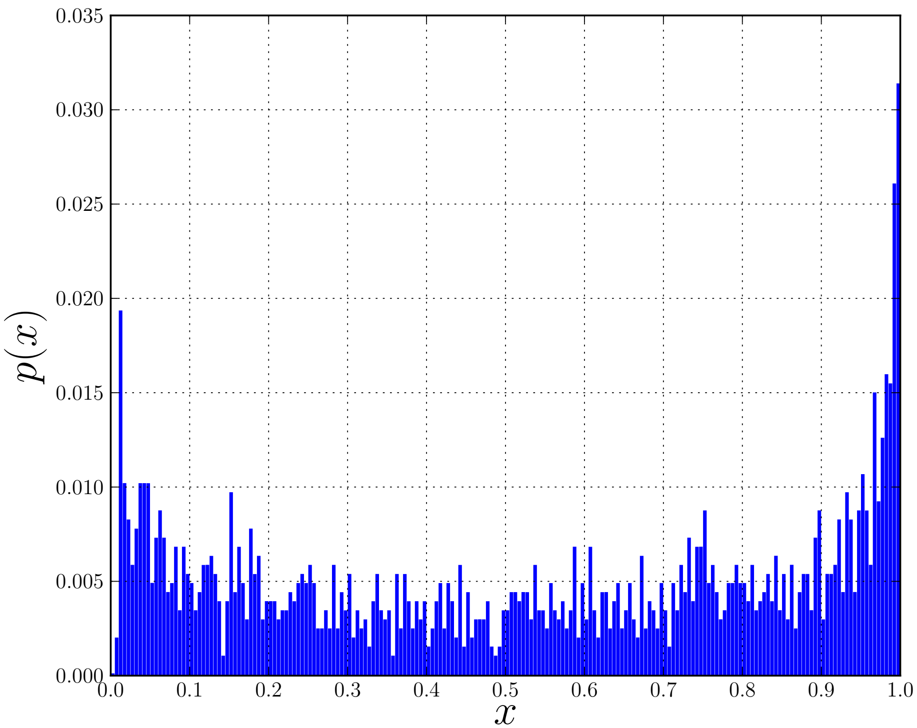


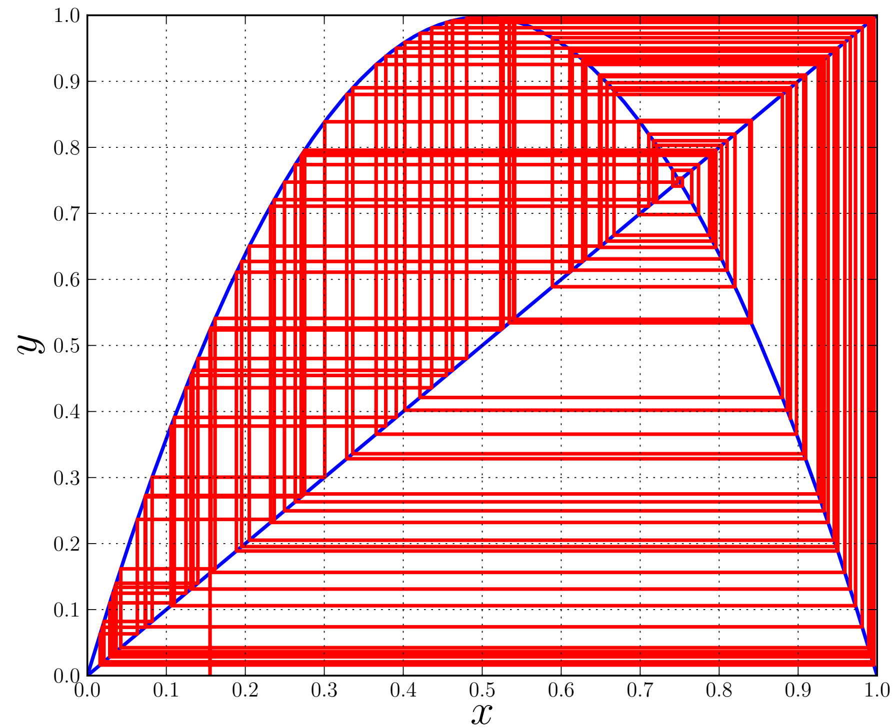
Percentage per interval Graphical interpretation.

$\lambda = 3.89$, $1,000,000$ iterations , 200 subintervals .

(The Figure on the right only shows 100 of these iterations.)

There is apparent *chaotic* behavior .





Percentage per interval Graphical interpretation.

$\lambda = 3.99$, $1,000,000$ iterations , 200 subintervals .

(The Figure on the right only shows 100 of these iterations.)

There is apparent *chaotic* behavior .

CONCLUSIONS :

- The behavior of the logistic equation depends on λ .
- For certain values of λ we see *fixed points* .
- For other values of λ there are *cycles* .
- For yet other values of λ there is seemingly *random behavior* .
- Many other deterministic equations have "*complex behavior*" !
- Nature is complex !

Generating Random Numbers

The logistic equation is a *recurrence relation* of the form

$$x_{k+1} \quad = \quad f(x_k) \ , \qquad k = 1, 2, 3, \ \cdots \ .$$

Most random number generators also have this form.

For the logistic equation we did *not* see sequences $\{x_k\}_{k=1}^{N}$ having

- a uniform distribution,
- a normal distribution,
- another known distribution.

QUESTION :

- How to generate *uniform* (and other) random numbers ?
- (These are useful in *computer simulations* .)

Generating Uniformly Distributed Random Numbers

Uniformly distributed random numbers can also be generated by a *recurrence relation* of the form

$$x_{k+1} \quad = \quad f(x_k) \ , \quad k = 1, 2, 3, \ \cdots \ .$$

Unlike the logistic equation, the x_k are most often *integers* .

The recurrence relation typically has the form

$$x_{k+1} \quad = \quad (n \ x_k) \bmod p \ .$$

where p is a *large prime number* , and n a *large integer* , with

$$p \quad \not| \ n \quad .$$

The following fact is useful :

THEOREM :

Let p be a prime number, and n an integer such that

$$p \not| \; n \; .$$

Then the function

$$f \quad : \quad \{ \; 0, 1, 2, \cdots, p-1 \; \} \quad \rightarrow \quad \{ \; 0, 1, 2, \cdots, p-1 \; \} \; ,$$

given by

$$f(x) \; = \; (n \; x) \bmod p \; ,$$

is *one-to-one* (and hence *onto*, a *bijection*, and *invertible*).

$$\boxed{p \ \text{prime}\ , \quad p \not| \ n \quad \Rightarrow \quad (n\ x) \bmod p \quad \text{is} \quad 1-1}$$

EXAMPLE :

$$p \ = \ 7 \quad \text{and} \quad n \ = \ 12 \ .$$

x	12x	12x mod 7
0	0	0
1	12	5
2	24	3
3	36	1
4	48	6
5	60	4
6	72	2

Invertible !

NOTE : The values of $12x \bmod 7$ look somewhat *random* !

$$\boxed{p \ \text{ prime}\ , \quad p \not| \ n \quad \Rightarrow \quad (n\ x) \bmod p \quad \text{is} \ \ 1-1}$$

EXAMPLE :

$$p \ = \ 6 \quad \text{and} \quad n \ = \ 2\ .$$

x	2x	2x mod 6
0	0	0
1	2	2
2	4	4
3	6	0
4	8	2
5	10	4

Not Invertible .

$$\boxed{p \ \text{ prime}\ , \quad p \not| \ n \quad \Rightarrow \quad (n\ x) \bmod p \quad \text{is} \quad 1-1}$$

EXAMPLE :

$$p = 6 \quad \text{and} \quad n = 13 \ .$$

x	13x	13x mod 6
0	0	0
1	13	1
2	26	2
3	39	3
4	52	4
5	65	5

Invertible . (So ?)

NOTE : The numbers in the right hand column don't look *random* !

$$\boxed{\quad p \ \text{ prime} \ , \quad p \ \not| \ n \quad \Rightarrow \quad (n\ x) \bmod p \quad \text{is} \ \ 1-1 \quad}$$

PROOF : *By contradiction* : Suppose the function is *not* $1-1$.

Then there are *distinct integers* x_1 , $x_2 \in \{ 0, 1, 2, \cdots, p-1 \}$, such that

$$(nx_1) \bmod p \ = \ k \qquad \text{and} \qquad (nx_2) \bmod p \ = \ k \ ,$$

where also $k \in \{ 0, 1, 2, \cdots, p-1 \}$. It follows that

$$p \ | \ n(x_1 - x_2) \ , \qquad (\text{ Why ? })$$

Since p is prime and $p \not| \ n$ it follows that

$$p \ | \ (x_1 - x_2) \ .$$

Thus $x_1 - x_2 \ = \ 0$ (Why ?) , *i.e.* , $x_1 \ = \ x_2$. Contradiction !

For given x_0 , an iteration of the form

$$x_k \; = \; (n\, x_{k-1}) \bmod p \; , \qquad k \; = \; 1, 2, \; \cdots \; , p-1 \; .$$

can be used to generate *random numbers* .

- Here p is a *large prime number* .
- The value of n is also *large* .
- The integer n must *not* be divisible by p .
- Do not start with $x_0 = 0$ (because it is a *fixed point* !) .
- Be aware of *cycles* (of period less than $p-1$) !

REMARK : Actually, *more often used* is an iteration of the form

$$x_k \; = \; (n\, x_{k-1} + m) \bmod p \; , \qquad k \; = \; 1, 2, \; \cdots \; , p-1 \; .$$

EXAMPLE : As a simple example, take again $p = 7$ and $n = 12$:

x	12x	12x mod 7
0	0	0
1	12	5
2	24	3
3	36	1
4	48	6
5	60	4
6	72	2

With $x_1 = 1$ the recurrence relation

$$x_{k+1} \quad = \quad f(x_k) \ , \qquad \text{where} \quad f(x) \ = \ 12x \bmod 7 \ ,$$

generates the sequence

$$1 \ \rightarrow \ 5 \ \rightarrow \ 4 \ \rightarrow \ 6 \ \rightarrow \ 2 \ \rightarrow \ 3 \ \rightarrow \ 1 \ \rightarrow \ 5 \ \rightarrow \ \cdots$$

which is *a cycle of maximal period* $p - 1$ (here 6) .

$f(x)$	sequence	cycle period
$5x \bmod 7$	$1 \to 5 \to 4 \to 6 \to 2 \to 3 \to 1$	**6**
$6x \bmod 7$	$1 \to 6 \to 1 \to 6 \to 1 \to 6 \to 1$	**2**
$8x \bmod 7$	$1 \to 1 \to 1 \to 1 \to 1 \to 1 \to 1$	**1**
$9x \bmod 7$	$1 \to 2 \to 4 \to 1 \to 2 \to 4 \to 1$	**3**
$10x \bmod 7$	$1 \to 3 \to 2 \to 6 \to 4 \to 5 \to 1$	**6**
$11x \bmod 7$	$1 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1$	**3**
$12x \bmod 7$	$1 \to 5 \to 4 \to 6 \to 2 \to 3 \to 1$	**6**

EXAMPLE : With $x_0 = 2$, compute

$$x_k \ = \ (137951\ x_{k-1}) \bmod 101\ , \qquad k \ = \ 1, 2, \ \cdots \ , 100\ .$$

Result :

71	46	17	48	88	94	4	41	92	34
96	75	87	8	82	83	68	91	49	73
16	63	65	35	81	98	45	32	25	29
70	61	95	90	64	50	58	39	21	89
79	27	100	15	78	42	77	57	54	99
30	55	84	53	13	7	97	60	9	67
5	26	14	93	19	18	33	10	52	28
85	38	36	66	20	3	56	69	76	72
31	40	6	11	37	51	43	62	80	12
22	74	1	86	23	59	24	44	47	2

QUESTION : Are there *repetitions* (*i.e.*, *cycles*) ?

EXAMPLE : As in the preceding example, use $x_0 = 2$, and compute

$$x_k = (137951\, x_{k-1}) \bmod 101 \,, \qquad k = 1, 2, \cdots, 100 \,,$$

and set

$$\hat{x}_k = \frac{x_k}{100} \,.$$

0.710	0.460	0.170	0.480	0.880	0.940	0.040	0.410	0.920	0.340
0.960	0.750	0.870	0.080	0.820	0.830	0.680	0.910	0.490	0.730
0.160	0.630	0.650	0.350	0.810	0.980	0.450	0.320	0.250	0.290
0.700	0.610	0.950	0.900	0.640	0.500	0.580	0.390	0.210	0.890
0.790	0.270	1.000	0.150	0.780	0.420	0.770	0.570	0.540	0.990
0.300	0.550	0.840	0.530	0.130	0.070	0.970	0.600	0.090	0.670
0.050	0.260	0.140	0.930	0.190	0.180	0.330	0.100	0.520	0.280
0.850	0.380	0.360	0.660	0.200	0.030	0.560	0.690	0.760	0.720
0.310	0.400	0.060	0.110	0.370	0.510	0.430	0.620	0.800	0.120
0.220	0.740	0.010	0.860	0.230	0.590	0.240	0.440	0.470	0.020

QUESTION : Do these numbers look *uniformly distributed* ?

EXAMPLE : With $x_0 = \mathbf{2}$, compute

$$x_k \;=\; (13795\mathbf{3}\; x_{k-1}) \bmod 101\ , \qquad k \;=\; 1, 2, \;\cdots\; , 100\ .$$

Result :

75	35	50	57	67	38	11	59	41	73
61	15	7	10	72	74	48	83	32	89
55	93	3	62	2	75	35	50	57	67
38	11	59	41	73	61	15	7	10	72
74	48	83	32	89	55	93	3	62	2
75	35	50	57	67	38	11	59	41	73
61	15	7	10	72	74	48	83	32	89
55	93	3	62	2	75	35	50	57	67
38	11	59	41	73	61	15	7	10	72
74	48	83	32	89	55	93	3	62	2

QUESTION : Are there *cycles* ?

EXAMPLE : With $x_0 = \mathbf{4}$, compute

$$x_k \;=\; (137953\, x_{k-1}) \bmod 101 \;, \qquad k \;=\; 1, 2, \;\cdots\; , 100 \;.$$

Result :

49	70	100	13	33	76	22	17	82	45
21	30	14	20	43	47	96	65	64	77
9	85	6	23	4	49	70	100	13	33
76	22	17	82	45	21	30	14	20	43
47	96	65	64	77	9	85	6	23	4
49	70	100	13	33	76	22	17	82	45
21	30	14	20	43	47	96	65	64	77
9	85	6	23	4	49	70	100	13	33
76	22	17	82	45	21	30	14	20	43
47	96	65	64	77	9	85	6	23	4

QUESTIONS :

- Are there *cycles* ?
- Is this the *same cycle* that we already found ?

Generating Random Numbers using the Inverse Method

- There are algorithms that generate *uniform* random numbers.

- These can be used to generate *other* random numbers.

- A simple method to do this is the *Inverse Transform Method* .

RECALL :

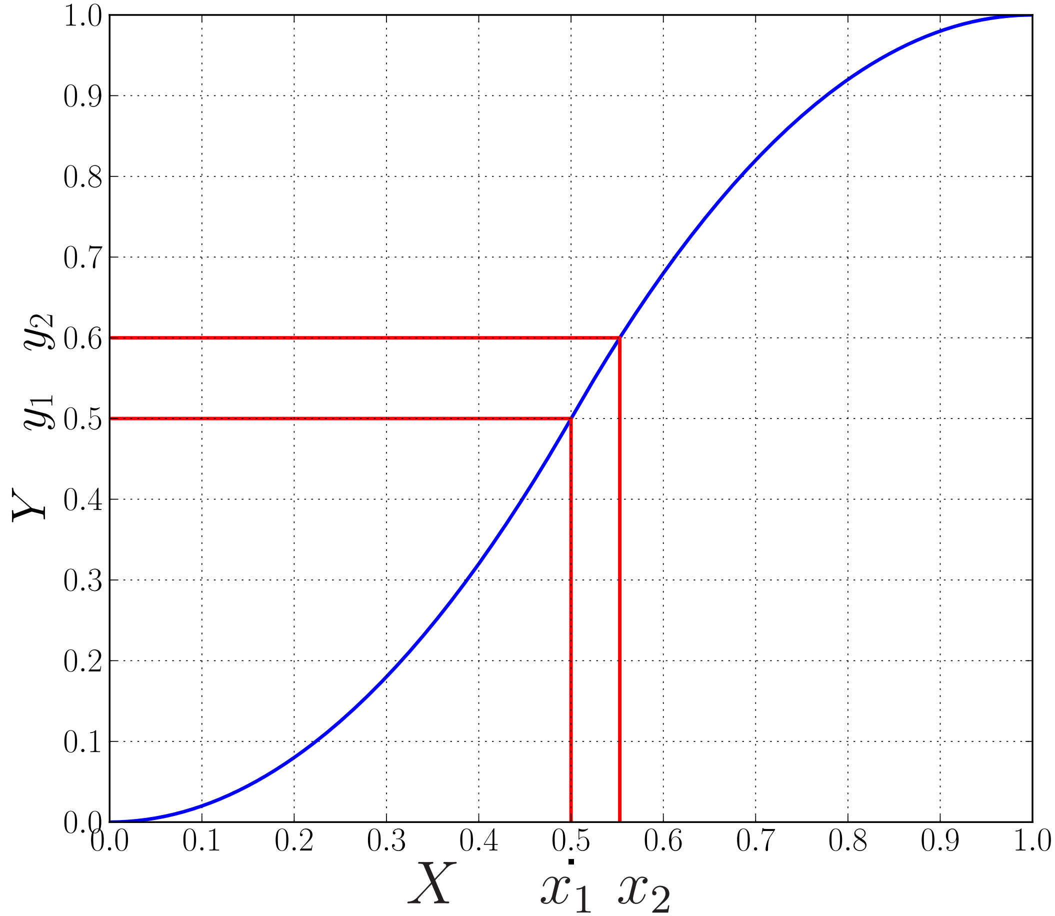
Let $f(x)$ be a *density function* on an interval $[a, b]$.

The *distribution function* is

$$F(x) \quad \equiv \quad \int_a^x f(x) \, dx \ .$$

- Since $f(x) \geq 0$ we know that $F(x)$ is *increasing* .

- If $F(x)$ is *strictly increasing* then $F(x)$ is *invertible* .



We want random numbers X with distribution $F(x)$ (blue) .

Let the random variable Y be uniform on the interval $[0,1]$.

$$\text{Let} \quad X = F^{-1}(Y) \ .$$

$$\text{Then} \quad P(x_1 \leq X \leq x_2) = y_2 - y_1 = F(x_2) - F(x_1) \ .$$

Thus $F(x)$ is indeed the distribution function of X !

- If Y is uniformly distributed on $[0, 1]$ then

$$P(\ y_1 \leq Y \leq y_2\ ) \quad = \quad y_2 - y_1\ .$$

- Let

$$X \quad = \quad F^{-1}(Y)\ ,$$

  with

$$x_1 \ = \ F^{-1}(y_1) \qquad \text{and} \qquad x_2 \ = \ F^{-1}(y_2)\ .$$

- Then

$$P(x_1 \leq X \leq x_2) \ = \ y_2 - y_1 \ = \ F(x_2) - F(x_1)\ . \qquad (\text{ Why ? })$$

- Thus $F(X)$ is also the *distribution function* of $X \equiv F^{-1}(Y)$!

NOTE : In the illustration X is on $[0, 1]$, but this is not necessary.

EXAMPLE :

Recall that the *exponential density function*

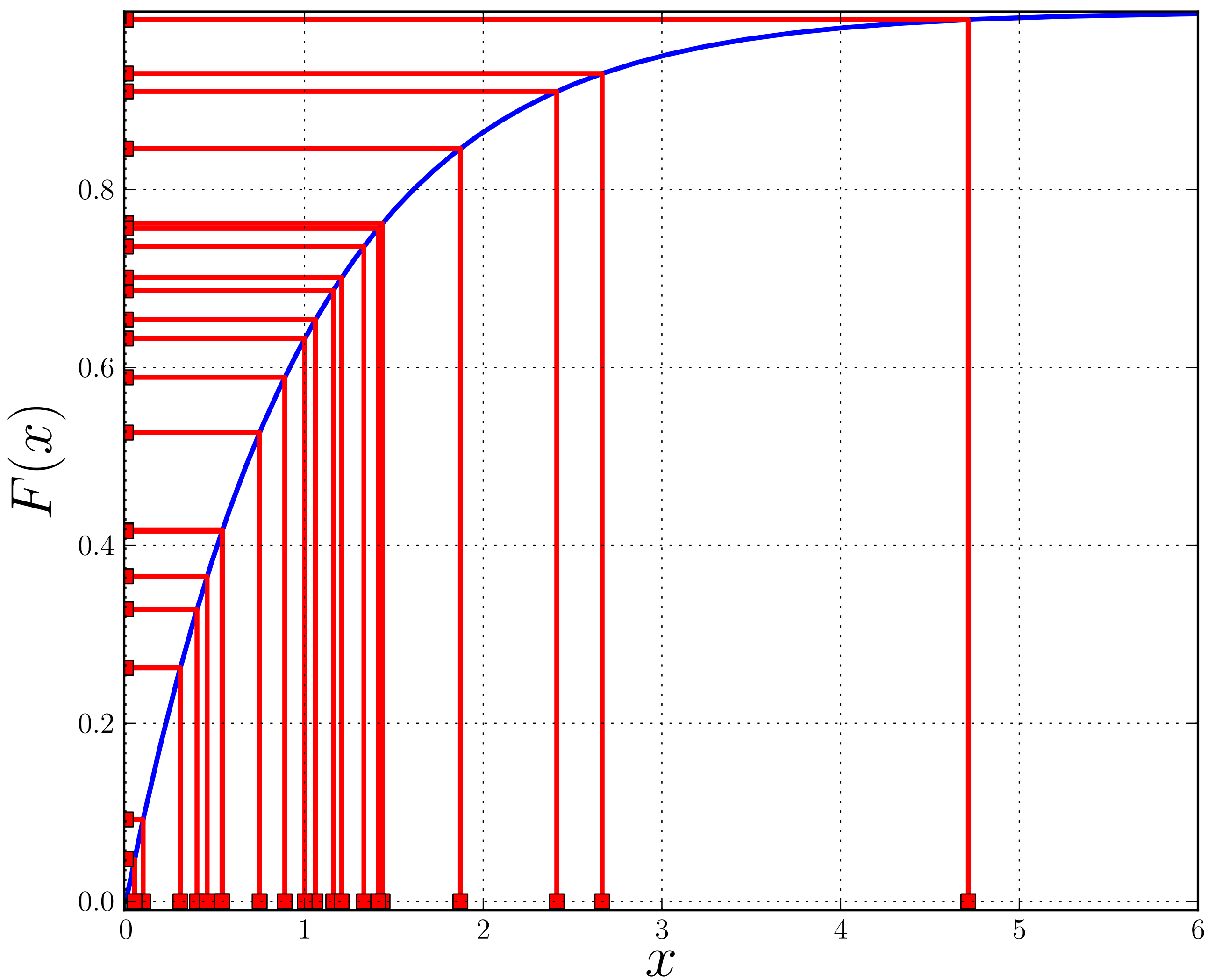
$$f(x) \;=\; \begin{cases} \lambda e^{-\lambda x} \;, & x > 0 \\ \\ 0 \;, & x \leq 0 \end{cases}$$

has *distribution function*

$$F(x) \;=\; \begin{cases} 1 - e^{-\lambda x} \;, & x > 0 \\ \\ 0 \;, & x \leq 0 \end{cases}$$

The *inverse distribution function* is

$$F^{-1}(y) \;=\; -\frac{1}{\lambda} \log(1-y) \;, \qquad 0 \leq y < 1 \;. \qquad (\text{ Check ! })$$



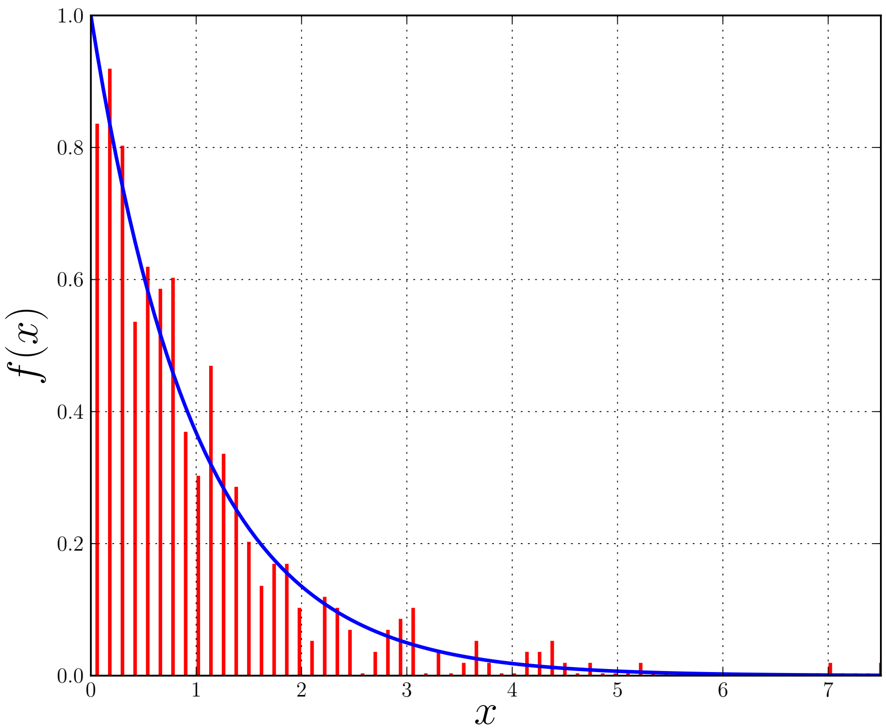
The inverse method for generating 20 " *exponential random numbers* ",

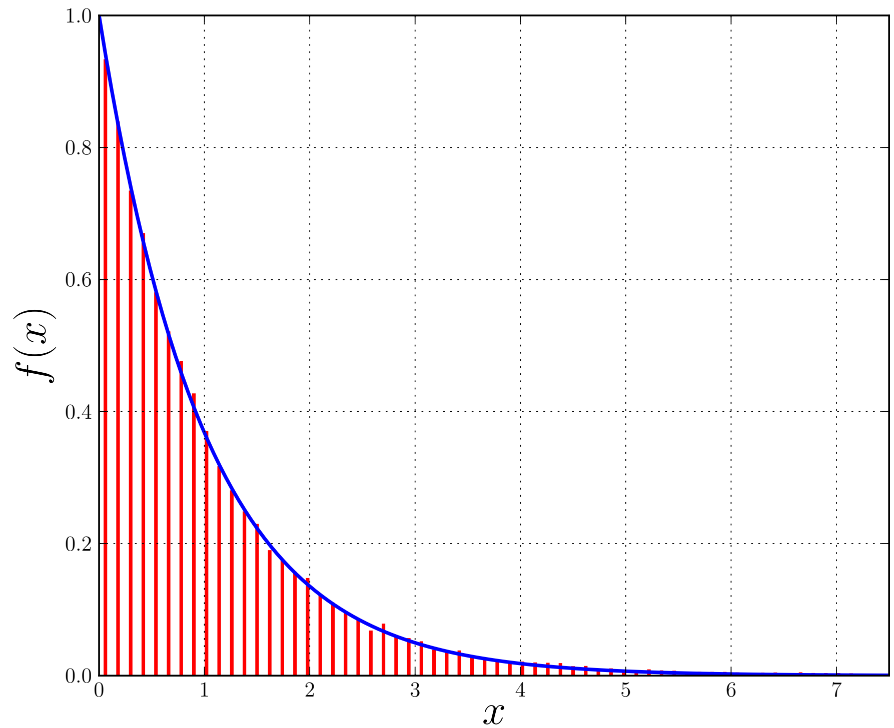
for the *exponential distribution function* with $\lambda = 1$.

EXAMPLE : (continued $\cdots$)

(Using the *inverse method* to get *exponential* random numbers.)

- Divide $[0,12]$ into 100 subintervals of equal size $\Delta x = 0.12$.
- Let I_k denote the kth interval, with midpoint x_k .
- Use the *inverse method* to get N *exponential* random numbers.
- Let m_k be the *frequency count* (# of random values in I_k) .
- Let $$\hat{f}(x_k) \;=\; \frac{m_k}{N\ \Delta x} \ .$$
- Then $$\int_0^\infty \hat{f}(x)\ dx \;\cong\; \sum_{k=1}^{100} \hat{f}(x_k)\ \Delta x \;=\; 1 \ .$$
- Then $\hat{f}(x)$ approximates the actual *density function* $f(x)$.





$N = 500$. $\qquad\qquad\qquad$ $N = 50,000$.

Simulating the exponential random variable with $\lambda = 1$.

(The actual density function is shown in *blue*.)

EXERCISE : Consider the *Tent density function*

$$f(x) \quad = \quad \begin{cases} x+1 \ , & -1 < x \leq 0 \\ 1-x \ , & 0 < x \leq 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

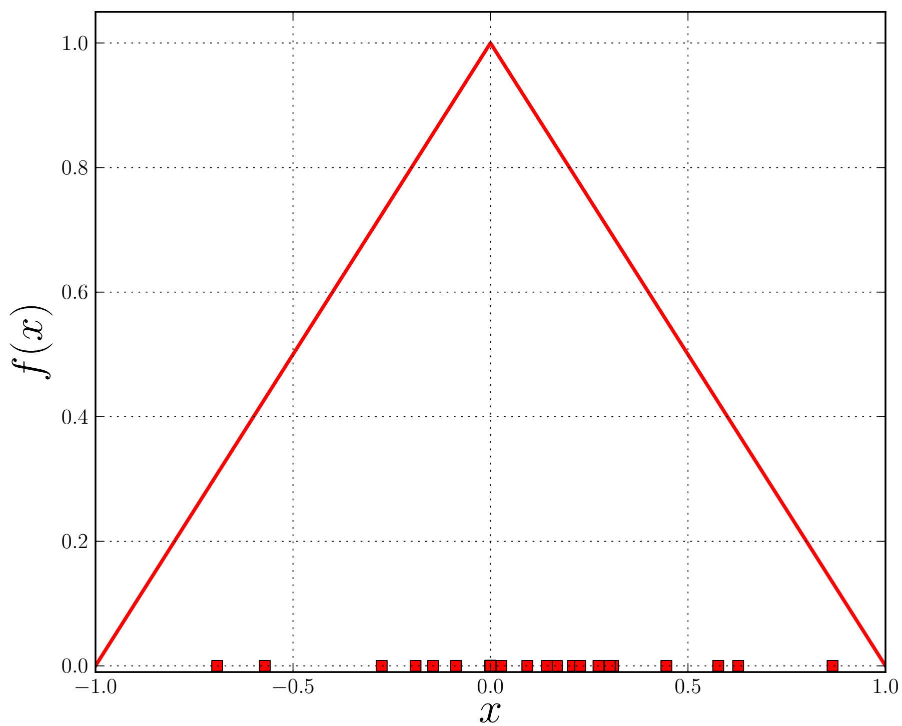
- Verify that that the distribution function is given by

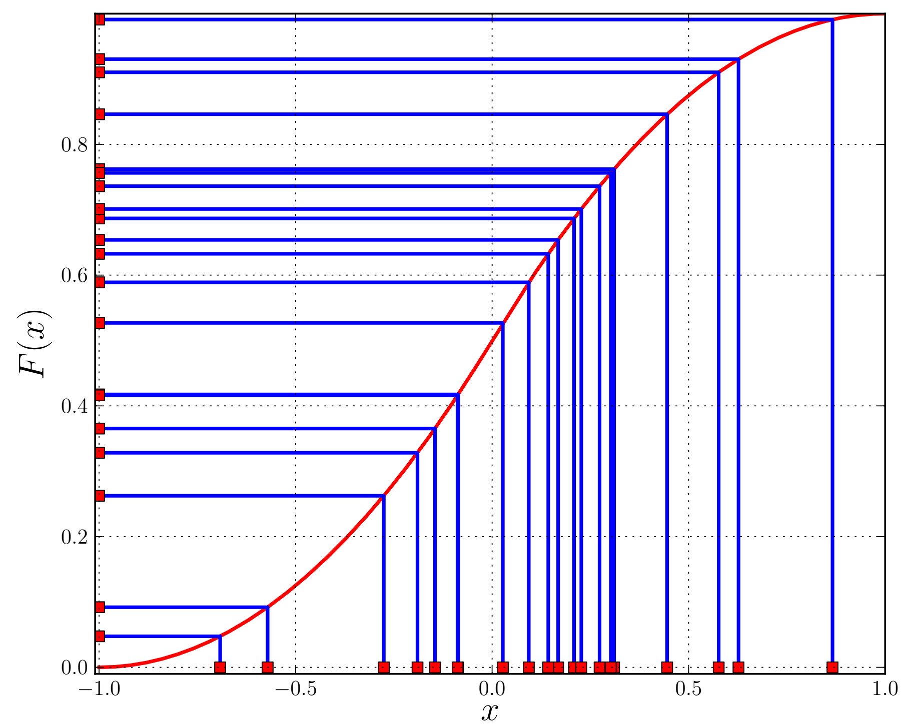
$$F(x) \quad = \quad \begin{cases} \frac{1}{2}x^2 + x + \frac{1}{2} \ , & -1 \leq x \leq 0 \\ \\ -\frac{1}{2}x^2 + x + \frac{1}{2} \ , & 0 < x \leq 1 \end{cases}$$

- Verify that that the *inverse* distribution function is

$$F^{-1}(y) \quad = \quad \begin{cases} -1 + \sqrt{2y} \ , & 0 \leq y \leq \frac{1}{2} \\ \\ 1 - \sqrt{2-2y} \ , & \frac{1}{2} < y \leq 1 \end{cases}$$

- Use the inverse method to generate " *Tent random numbers* ".

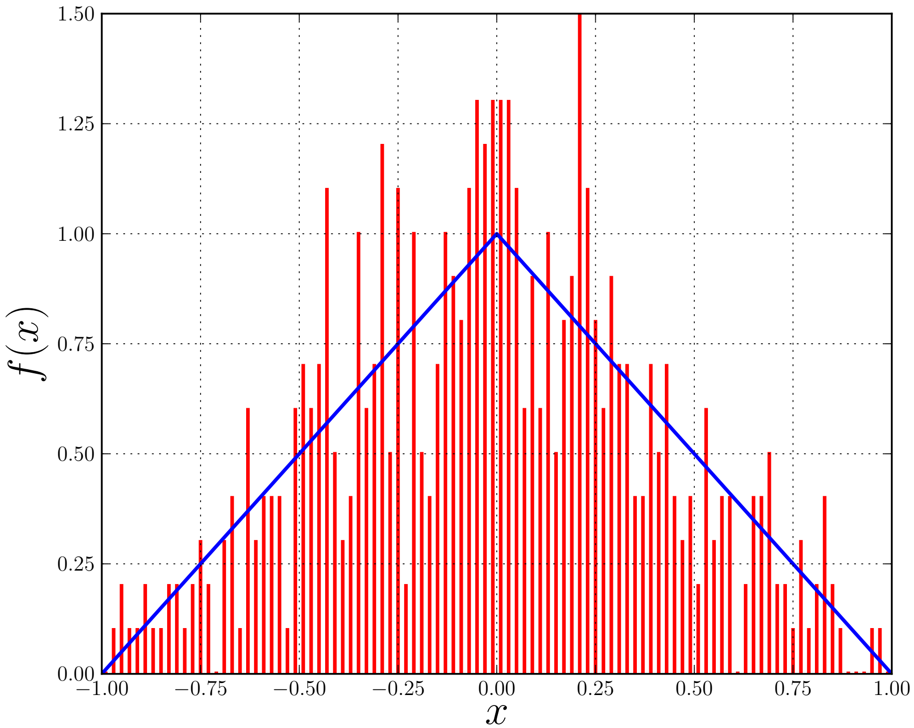


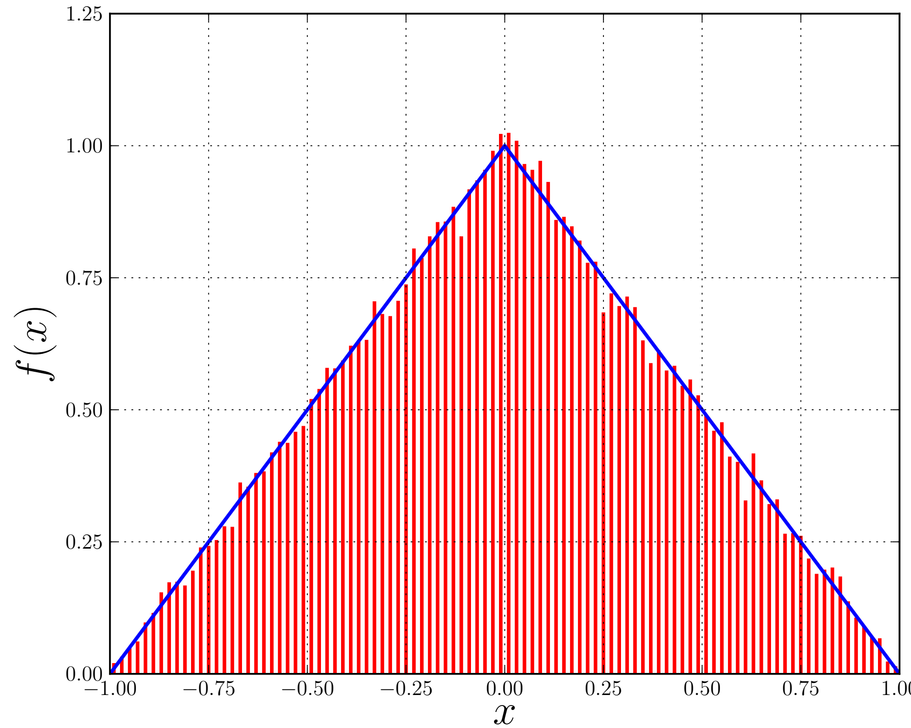


The "*Tent*" density Function.

The "*Tent*" distribution function.

The inverse method for generating 20 "*Tent random numbers*".





$N = 500$. $N = 50,000$.

Simulating the " *Tent random variable* " .

(The actual density function is shown in *blue*.)

SUMMARY TABLES

and

FORMULAS

Discrete	Continuous
$p(x_i) = P(X = x_i)$	$f(x)\delta x \ \cong \ P(x - \frac{\delta}{2} < X < x + \frac{\delta}{2})$
$\sum_i p(x_i) = 1$	$\int_{-\infty}^{\infty} f(x)\ dx = 1$
$F(x_k) = \sum_{i \leq k} p(x_i)$	$F(x) = \int_{-\infty}^{x} f(x)\ dx$
$p(x_k) = F(x_k) - F(x_{k-1})$	$f(x) = F'(x)$
$E[X] = \sum_i x_i p(x_i)$	$E[X] = \int_{-\infty}^{\infty} x\ f(x)\ dx$
$E[g(X)] = \sum_i g(x_i)p(x_i)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f(x)\ dx$
$E[XY] = \sum_{i,j} x_i y_j p(x_i, y_j)$	$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y)\ dy\ dx$

Name	**General Formula**
Mean	$\mu = E[X]$
Variance	$Var(X) \; = \; E[(X-\mu)^2] \; = \; E[X^2] - \mu^2$
Covariance	$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y$
Markov	$P(X \geq c) \; \leq \; E[X]/c$
Chebyshev	$P(\mid X - \mu \mid \geq k\sigma) \; \leq \; 1/k^2$
Moments	$\psi(t) = E[e^{tX}] \; , \; \psi'(0) = E[X] \; , \; \psi''(0) = E[X^2]$

Name	Probability mass function	Domain
Bernoulli	$P(X=1)=p\ ,\ \ P(X=0)=1-p$	$0\ ,\ 1$
Binomial	$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$	$0\le k\le n$
Poisson	$P(X=k)\ =\ e^{-\lambda}\ \frac{\lambda^k}{k!}$	$k=0,1,2,\cdots$

Name	**Mean**	**Standard deviation**
Bernoulli	p	$\sqrt{p(1-p)}$
Binomial	np	$\sqrt{np(1-p)}$
Poisson	λ	$\sqrt{\lambda}$

Name	**Density function**	**Distribution**	**Domain**
Uniform	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$x \in (a, b]$
Exponential	$\lambda e^{-\lambda x}$	$1 - e^{-\lambda x}$	$x \in (0, \infty)$
Std. Normal	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$		$x \in (-\infty, \infty)$
Normal	$\frac{1}{\sqrt{2\pi}\ \sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$		$x \in (-\infty, \infty)$

Name	**Mean**	**Standard Deviation**
Uniform	$\frac{a+b}{2}$	$\frac{b-a}{2\sqrt{3}}$
Exponential	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$
Standard Normal	0	1
General Normal	μ	σ
Chi-Square	n	$\sqrt{2n}$

The Standard Normal Distribution $\Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$
0.0	.5000	-1.2	.1151
-0.1	.4602	-1.4	.0808
-0.2	.4207	-1.6	.0548
-0.3	.3821	-1.8	.0359
-0.4	.3446	**-2.0**	**.0228**
-0.5	.3085	-2.2	.0139
-0.6	.2743	-2.4	.0082
-0.7	.2420	-2.6	.0047
-0.8	.2119	-2.8	.0026
-0.9	.1841	-3.0	.0013
-1.0	.1587	-3.2	.0007

(For example, $P(Z \leq \mathbf{-2.0}) = \Phi(\mathbf{-2.0}) = \mathbf{2.28}\%$)

The χ_n^2 - Table

n	$\alpha = 0.975$	$\alpha = \mathbf{0.95}$	$\alpha = 0.05$	$\alpha = 0.025$
5	0.83	1.15	11.07	12.83
6	1.24	1.64	12.59	14.45
7	1.69	2.17	14.07	16.01
8	2.18	2.73	15.51	17.54
9	2.70	3.33	16.92	19.02
10	3.25	**3.94**	18.31	20.48
11	3.82	4.58	19.68	21.92
12	4.40	5.23	21.03	23.34
13	5.01	5.89	22.36	24.74
14	5.63	6.57	23.69	26.12
15	6.26	7.26	25.00	27.49

This Table shows $z_{\alpha,n}$ values such that $P(\chi_n^2 \geq z_{\alpha,n}) = \alpha$.

(For example, $P(\chi_{\mathbf{10}}^2 \geq \mathbf{3.94}) = \mathbf{95}\%$)

The T - distribution Table

n	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha =$ **0.01**	$\alpha = 0.005$
5	-1.476	-2.015	-3.365	-4.032
6	-1.440	-1.943	-3.143	-3.707
7	-1.415	-1.895	-2.998	-3.499
8	-1.397	-1.860	-2.896	-3.355
9	-1.383	-1.833	-2.821	-3.250
10	-1.372	-1.812	**-2.764**	-3.169
11	-1.363	-1.796	-2.718	-3.106
12	-1.356	-1.782	-2.681	-3.055
13	-1.350	-1.771	-2.650	-3.012
14	-1.345	-1.761	-2.624	-2.977
15	-1.341	-1.753	-2.602	-2.947

This Table shows $t_{\alpha,n}$ values such that $P(T_n \leq t_{\alpha,n}) = \alpha$.

(For example, $P(T_{\mathbf{10}} \leq \mathbf{-2.764}) = \mathbf{1}\%$)