BINOMIAL DISTRIBUTION

Characteristics of a binomial distribution:-

(i) An outcome on each trial of an experiment is classified into one of two mutually exclusive categories- a success or a failure.
(ii) The random variable counts the number of success in a fixed number of trials.
(iii) The probability of success and failure stays the same for each trial.
(iv) The trials are independent, meaning that the outcome of one trial does not affect the outcome of any other trial.

Probability function:-

The binomial probability distribution is computed by the formula:

$$p(X = x) = \begin{cases} \left({}^{n}C_{x}\right) p^{x} (1-p)^{n-x};\ x = 0, 1, 2, \text{.........}, n \\ 0;\ otherwise \end{cases}$$

where, *n* is the number of trials.
X is the random variable defined as the number of success.
p is the probability of a success on each trial.

Mean and variance:- The mean (μ) and the variance (σ^2) of a binomial distribution can be computed in a 'shortcut' fashion by:

$$\mu = E(X) = np$$
$$\sigma^2 = E(X-\mu)^2 = np(1-p)$$

Example:- There are five flights daily from Pittsburg via Us Airways into the Bradford, Pennsylvania Regional Airport. Suppose the probability that any flight arrives late is 0.20. What is the probability that none of the flights are late today? What is the probability that exactly one of the flights is late today? Also find the mean and variance.

Solution:- Let *X* denotes the number of plane that arrives late.

For this problem, $n = 5$ and $p = 0.20$.

The probability that no flights will arrive late today is

$$\begin{aligned} p(X = 0) &= \left({}^{5}C_{0}\right) \times 0.2^{0} \times (1-0.2)^{5-0} \\ &= \left({}^{5}C_{0}\right) \times 0.2^{0} \times 0.8^{5} \\ &= 0.3277 \ \textbf{(Ans.)} \end{aligned}$$

The probability that exactly one flight will arrive late today is

$$p(X = 1) = \left({}^{5}C_{1}\right) \times 0.2^{1} \times (1-0.2)^{5-1}$$

$$= \left({}^{5}C_{1}\right)\times 0.2^{1}\times 0.8^{4}$$
$$= 0.4096 \textbf{ (Ans.)}$$

Now, mean, $\mu = np$
$$= 5\times 0.20$$
$$= 1.0 \textbf{ (Ans.)}$$

Variance, $\sigma^{2} = np(1-p)$
$$= 5\times 0.20\times(1-0.20)$$
$$= 0.80 \textbf{ (Ans.)}$$

Example:- The probability that a certain kind of component will survive a shock test is $\frac{3}{4}$. Find the probability that exactly 2 of the next 4 components tested survive.

Solution:- Let X denotes the number of components that will survive a shock test.

For this problem, $n = 4$, $p = \frac{3}{4}$.

$$\because\ p(X = 2) = \left({}^{4}C_{2}\right)\times 0.75^{2}\times(1-0.75)^{4-2}$$
$$= \left({}^{4}C_{2}\right)\times 0.75^{2}\times 0.25^{2}$$
$$= 0.2109375 \textbf{(Ans.)}$$

Example:- The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 13 survive, (b) from 3 to 5 survive, and (c) exactly 2 survive?

Solution:- Let X be the number of people that survive.

For this problem, $n = 15$, $p = 0.4$.

(a) $p(X \geq 13) = p(X = 13) + p(X = 14) + p(X = 15)$
$$= \left({}^{15}C_{13}\right)\times 0.4^{13}\times 0.6^{2} + \left({}^{15}C_{14}\right)\times 0.4^{14}\times 0.6^{1} + \left({}^{15}C_{15}\right)\times 0.4^{15}\times 0.6^{0}$$
$$= 0.000278904 \textbf{ (Ans.)}$$

(b) $p(3 \leq X \leq 5) = p(X = 3) + p(X = 4) + p(X = 5)$
$$= \left({}^{15}C_{3}\right)\times 0.4^{3}\times 0.6^{12} + \left({}^{15}C_{4}\right)\times 0.4^{4}\times 0.6^{11} + \left({}^{15}C_{5}\right)\times 0.4^{5}\times 0.6^{10}$$
$$= 0.376101549 \textbf{ (Ans.)}$$

(c) $p(X = 2) = \left({}^{15}C_{2}\right)\times 0.4^{2}\times 0.6^{13}$
$$= 0.021941965 \textbf{ (Ans.)}$$

Example:- Five percent of the worm gears produced by an automatic, high-speed Carter-Bell milling machine are defective. What is the probability that out of six gears selected at random none will be defective? Exactly six out of six? Also find the mean and variance.

Solution:- Let X denotes the number of defective gears.

For this problem, $n = 6$, $p = 0.05$.

The probability that none of the gears will be defective is

$$
\begin{aligned}
p(X = 0) &= \left({}^{6}C_{0}\right) \times 0.05^{0} \times (1 - 0.05)^{6-0} \\
&= \left({}^{6}C_{0}\right) \times 0.05^{0} \times 0.95^{6} \\
&= 0.735 \textbf{ (Ans.)}
\end{aligned}
$$

The probability that exactly six gears will be defective is

$$
\begin{aligned}
p(X = 6) &= \left({}^{6}C_{6}\right) \times 0.05^{6} \times (1 - 0.05)^{6-6} \\
&= \left({}^{6}C_{6}\right) \times 0.05^{6} \times 0.95^{0} \\
&= 0.000 \textbf{ (Ans.)}
\end{aligned}
$$

Mean,
$$
\begin{aligned}
\mu &= np \\
&= 6 \times 0.05 \\
&= 0.30 \textbf{ (Ans.)}
\end{aligned}
$$

Variance,
$$
\begin{aligned}
\sigma^{2} &= np(1 - p) \\
&= 6 \times 0.05 \times (1 - 0.05) \\
&= 6 \times 0.05 \times 0.95 \\
&= 0.285 \textbf{ (Ans.)}
\end{aligned}
$$

Example:- A large chain retailer purchase a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 3%. The inspector of the retailer randomly picks 20 items from a shipment. What is the probability that there will be at least one defective item among these 20?

Solution:- Let X denotes the number of defective devices.

For this problem $n = 20$ and $p = 0.03$.

$$
\begin{aligned}
\because \; p(X \geq 1) &= 1 - p(X = 0) \\
&= 1 - \left({}^{20}C_{0}\right) \times 0.03^{0} \times 0.97^{20} \\
&= 0.4562 \textbf{ (Ans.)}
\end{aligned}
$$

Example:- A recent study by the American Highway Patrolman's Association revealed that 60 percent of American drivers use seat belts. A sample of 10 drivers on the Florida Turnpike is selected.

(i) What is the probability that exactly 7 are wearing seat belts?

(ii) What is the probability that 7 or fewer of the drivers are wearing seat belts?

Solution:- Let X denotes the number of drivers wearing seat belts.

For this problem, $n = 10$, $p = 0.60$.

(i) The probability that exactly 7 are wearing seat belts is

$$\begin{aligned} p(X = 7) &= \left({}^{10}C_7\right) \times 0.6 \times (1 - 0.6)^{10-7} \\ &= \left({}^{10}C_7\right) \times 0.6^7 \times 0.4^3 \\ &= 0.215 \textbf{ (Ans.)} \end{aligned}$$

(ii) The probability that 7 or fewer of the drivers are wearing seat belts is

$$\begin{aligned} p(X \le 7) &= 1 - p(X > 7) \\ &= 1 - \left[p(X = 8) + p(X = 9) + p(X = 10)\right] \\ &= 1 - \left[\left({}^{10}C_8\right) \times 0.6^8 \times 0.4^2 + \left({}^{10}C_9\right) \times 0.6^9 \times 0.4^1 + \left({}^{10}C_{10}\right) \times 0.6^{10} \times 0.4^0\right] \\ &= 1 - 0.167 \\ &= 0.833 \textbf{ (Ans.)} \end{aligned}$$

POISSON DISTRIBUTION

The Poisson distribution describes the number of times some event occurring during a specified interval. The interval may be time, distance, area, or, volume.

This distribution is a limiting form of the binomial distribution when the probability of a success is very small and *n* is large.

Assumptions:- The assumptions of a Poisson probability distribution are:

(i) The probability of the occurrence of an event is constant for all subintervals.
(ii) There can be no more than one occurrence in each subinterval.
(iii) Occurrences are independent, that is, the number of occurrences in any non-overlapping intervals is independent of one another.

Probability function:-

The Poisson distribution can be described mathematically by the formula:

$$p(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}\ ;\ x = 0, 1, 2, ..., \infty$$

where, λ is the mean number of occurrences (successes) in a particular interval.
x is the number of occurrences (successes).
$p(X = x)$ is the probability for a specified value of X.

Mean and variance:- The mean and variance of the Poisson probability distribution are:

$$\mu = E(X) = \lambda = np \quad \sigma^2 = \lambda = np$$

That is, mean and variance of a Poisson distribution are equal.

The Poisson distribution as a limiting form of the Binomial:

i. *p*, the probability of success in a Bernoulli trial is very small, i.e., $p \to 0$.
ii. *n*, the number of trials is very large, i.e., $n \to \infty$
iii. $np = \lambda$ is finite constant, that is average number of success is finite.

Example:- During a laboratory experiment the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Solution:- Let *X* denotes the number of particles.

For this problem $\lambda = 4$.

$$\therefore \; p(X = 6) = \frac{e^{-4} 4^6}{6!} = 0.1042$$

Example:- If the probability that a car accident happens in a very busy road in an hour is 0.001. If 2000 cars passed in one hour by that road, what is the probability that (i) exactly 3, (ii) more than two car accidents happened on that hour of the road?

Solution:- Let *X* be the number of car accident which follow Poisson distribution with $\lambda = 2000 \times 0.001 = 2$, as the probability of accident is very small.

(i) $p(X = 3) = \dfrac{e^{-2} 2^3}{3!} = 0.180$

(ii) $p(X > 2) = 1 - p(X \le 2)$

$$= 1 - [p(X = 0) + p(X = 1) + p(X = 2)]$$

$$= 1 - \left[\frac{e^{-2} 2^0}{0!} + \frac{e^{-2} 2^1}{1!} + \frac{e^{-2} 2^2}{2!}\right]$$

$$= 0.323$$

Example:- In a manufacturing process where glass products are produced, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, one in every 1000 of these items produced have one or more bubbles. What is the probability that a random sample of 4000 will yield fewer than 2 items possessing bubbles?

Solution:- This is essentially a binomial experiment with $n = 4000$ and $p = 0.001$. Since *p* is very close to zero and *n* is quite large, we shall approximate with the Poisson distribution with $\lambda = 4000 \times 0.001 = 4$.

Hence, if *X* represents the number of bubbles, we have

$$p(X < 2) = p(X = 0) + p(X = 1)$$

$$= \frac{e^{-4} 4^0}{0!} + \frac{e^{-4} 4^1}{1!}$$

$$= 0.0916$$

Example:- Andrew Whittaker, computer centre manager, reports that his computer system experienced three component failures during the past 100 days.
(i) What is the probability of no failures in a given day?
(ii) What is the probability of one or more component failures in a given day?

Solution:-The expected number of failures per day is $\lambda = \frac{3}{100} = 0.03$.

(i) $p(no\ \ failures\, in\ a\ \ given\, day) = p(X = 0) = \frac{0.03^0 \times e^{-0.03}}{0!}$
$= 0.970446$ **(Ans.)**

(ii) $p(at\ least\ one\ component\ \ failure\, in\ a\ \ given\, day) = p(X \geq 1)$
$= 1 - p(X < 1)$
$= 1 - p(X = 0)$
$= 1 - \frac{0.03^0 \times e^{-0.03}}{0!}$
$= 1 - 0.970446$
$= 0.029554$ **(Ans.)**

NORMAL DISTRIBUTION

A continuous random variable *X* is said to have a normal distribution if its probability density function is given by

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \ ;\ -\infty < x < \infty$$

where the parameters μ and σ^2 satisfy $-\infty < \mu < \infty$, $\sigma^2 > 0$. The variable *X* whose density function is called normal density with parameters μ and σ^2 is denoted by $N(\mu,\sigma^2)$.

The parameters μ and σ^2 are actually the mean and variance of the normal variate *X*.

If *X* is a normal variate with parameters μ and σ^2, then $Z = \frac{X-\mu}{\sigma}$ is a standard normal variate with mean zero and variance unity.

A continuous random variable *Z* is said to have a standard normal distribution if its probability density function is given by

$$f(z;0,1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\ ;\ -\infty<z<\infty$$

which is parameter free distribution, since here μ and σ^2 are known.

Show that $Z=\frac{X-\mu}{\sigma}$ has mean 0 and variance 1.

Proof:- $E(Z)=E\left(\frac{X-\mu}{\sigma}\right)$

$$=E\left(\frac{X}{\sigma}-\frac{\mu}{\sigma}\right)$$

$$=\frac{1}{\sigma}E(X)-\frac{\mu}{\sigma}$$

$$=\frac{1}{\sigma}\mu-\frac{\mu}{\sigma}$$

$$=\frac{\mu}{\sigma}-\frac{\mu}{\sigma}$$

$$=0$$

$$V(Z)=V\left(\frac{X-\mu}{\sigma}\right)$$

$$=V\left(\frac{X}{\sigma}-\frac{\mu}{\sigma}\right)$$

$$=V\left(\frac{X}{\sigma}\right)$$

$$=\frac{1}{\sigma^2}V(X)$$

$$=\frac{1}{\sigma^2}\times\sigma^2$$

$$=1$$

Properties or characteristics of normal distribution:-

1. The curve of the distribution is symmetrical about the point $x=\mu$ and it is bell shaped.
2. For normal distribution mean, median and mode are same, which is equal to μ.
3. For normal distribution, skewness and kurtosis are $\beta_1=0$ and $\beta_2=3$.
4. Linear combination of independent normal variates is also a normal variate.

5. The curve approaches nearer and nearer to the base but it never touches it, i.e., the curve is asymptotic to the base on either side. Hence the ranges are unlimited or infinite in both directions.
6. Under certain condition most of the distribution tends to normal distribution.
7. The area under the normal curve is distributed as follows:
   (a) 68.26% of the time, a normal random variable assumes a value within plus or minas 1 standard deviation of its mean.
   (b) 95.44% of the time, a normal random variable assumes a value within plus or minas 2 standard deviation of its mean.
   (c) 99.72% of the time, a normal random variable assumes a value within plus or minas 3 standard deviation of its mean.

Example:- A client has an investment portfolio whose mean value is equal to \$500,000 with a standard deviation of \$15,000. She has asked you to determine the probability that the value of her portfolio is between \$485,000 and \$530,000.

Solution:- Let X denotes the investment portfolio.

$$\therefore\ p(485{,}000 < X < 530{,}000) = p\left(\frac{485{,}000-500{,}000}{15{,}000} < \frac{X-\mu}{\sigma} < \frac{530{,}000-500{,}000}{15{,}000}\right)$$

$$= p(-1 < Z < 2)$$

$$= p(-1 < Z < 0) + p(0 < Z < 2)$$

$$= 0.3413 + 0.4772$$

$$= 0.8185$$ (**Ans.**)

Example:- If $X \sim N(15, 16)$, find the probability that X is larger than 18.

Solution:- $p(X > 18) = p\left(\frac{X-\mu}{\sigma} > \frac{18-15}{4}\right)$

$$= p(Z > 0.75)$$

$$= p(0 < Z < \infty) - p(0 < Z < 0.75)$$

$$= 0.5 - 0.2734$$

$$= 0.2266$$ (**Ans.**)

Example:- A company produces lightbulbs whose life follows a normal distribution with mean 1,200 hours and standard deviation 250 hours. If we choose a lightbulb at random, what is the probability that its lifetime will be between 900 and 1,300 hours?

Solution:- Let X represents lifetime in hours. Then

$$p(900 < X < 1{,}300) = p\left(\frac{900-1{,}200}{250} < \frac{X-\mu}{\sigma} < \frac{1{,}300-1{,}200}{250}\right)$$

$$= p(-1.2 < Z < 0.4)$$

$$= p(-1.2 < Z < 0) + p(0 < Z < 0.4)$$

$$= 0.3849 + 0.1554$$

$= 0.5403$ (**Ans.**)

Example:- A very large group of students obtains test scores that are normally distributed with mean 60 and standard deviation 15. What proportion of students obtained scores between 85 and 95?

Solution:- Let X denotes the test scores. Then

$$p(85 < X < 95) = p\left(\frac{85-60}{15} < \frac{X-\mu}{\sigma} < \frac{95-60}{15}\right)$$

$$= p(1.67 < Z < 2.33)$$

$$= p(0 < Z < 2.33) - p(0 < Z < 1.67)$$

$$= 0.4901 - 0.4525$$

$= 0.0376$ (**Ans.**)