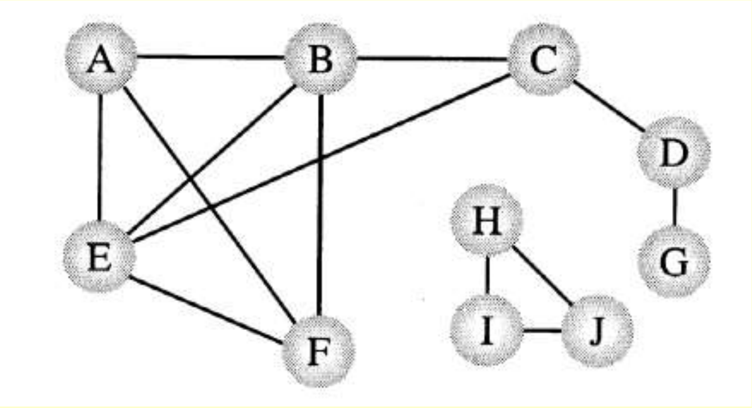
CSE301 – Combinatorial Optimization

Articulation Points, Bridges & Biconnected Components

Connectivity/Biconnectivity for Undirected Graph

A node and all the nodes reachable from it compose a **connected component**. A graph is called **connected** if it has only one connected component.

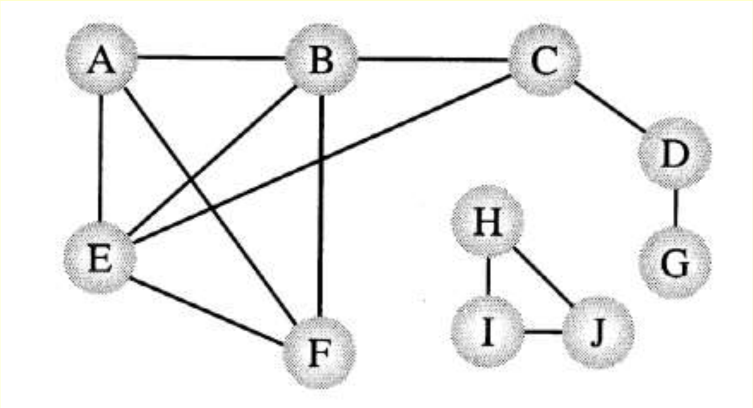
Since the function **visit**() of DFS visits every node that is reachable and has not already been visited, the DFS can easily be modified to print out the connected components of a graph.



Two connected components

Connectivity/Biconnectivity

In actual uses of graphs, such as networks, we need to establish not only that every node is connected to every other node, but also there are **at least two independent paths between any two nodes**. A maximum set of nodes for which there are two different paths is called **biconnected**.



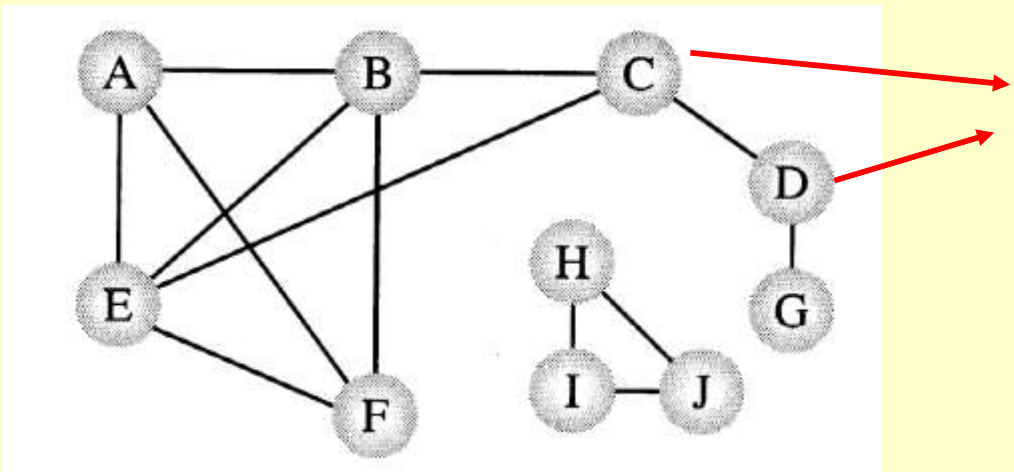
{H,I,J} and {A,B,C,E,F} are biconnected.

Connectivity/Biconnectivity

Another way to define this concept is that there are **no single points of failure**, no nodes that when deleted along with any adjoining arcs, would split the graph into two or more separate connected components. Such a node is called an **articulation point**.

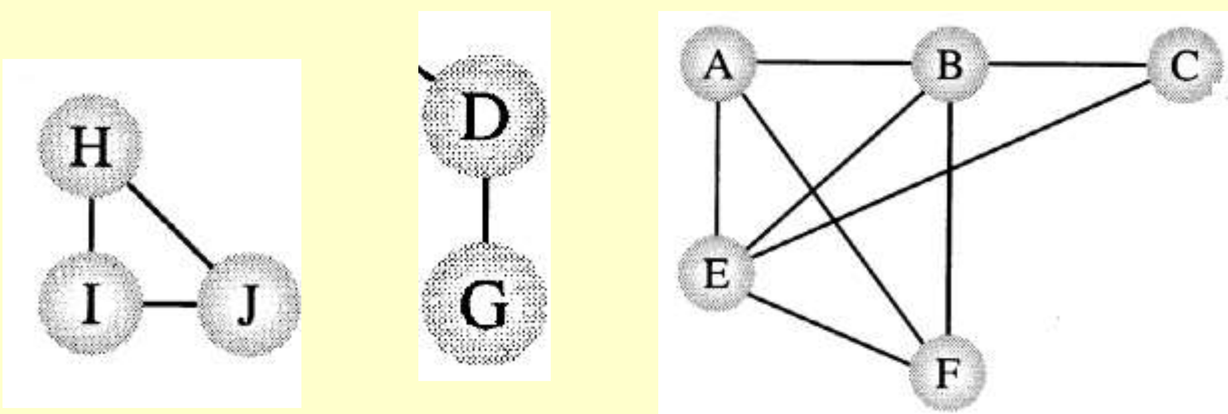
If a graph contains no articulation points, then it is biconnected. If a graph does contain articulation points, then it is useful to split the graph into the pieces where each piece is a maximal biconnected subgraph called a **biconnected component**.

Connectivity/Biconnectivity



Articulation points

Three biconnected components



Finding Articulations

- Problem:
  - Given any graph $G = (V, E)$, find all the articulation points.
  - Possible strategy:
    - For all vertices v in V:

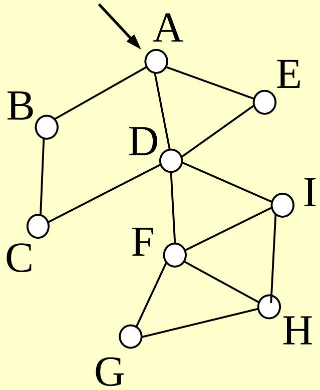
      Remove v and its incident edges

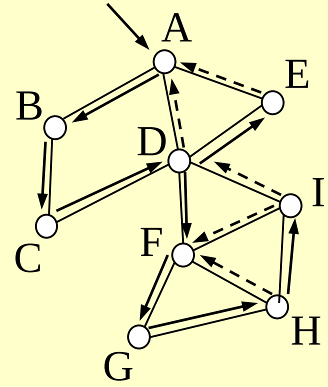
      Test connectivity using a DFS.
    - Execution time: $\Theta(n(n+m))$.

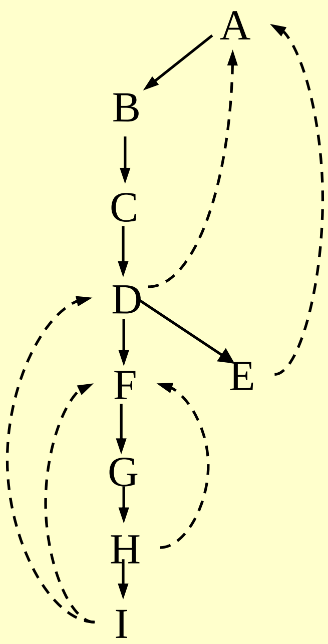
- Can we do better?

Finding Articulation Points

- A DFS tree can be used to discover articulation points in $\Theta(n + m)$ time.

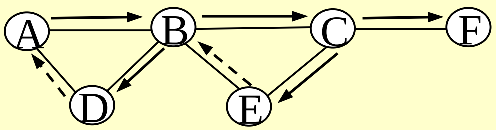




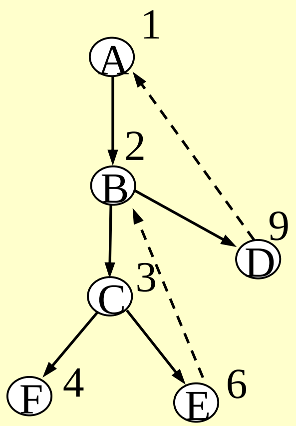


Can you characterize *D* ?

Depth First Search number

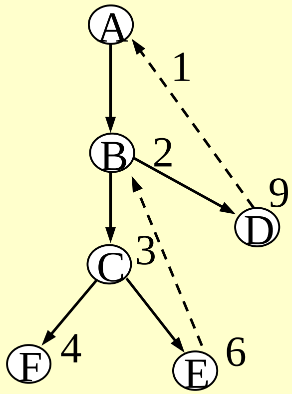


$G = (V, E)$



A	B	C	D	E	F
1	2	3	9	6	4

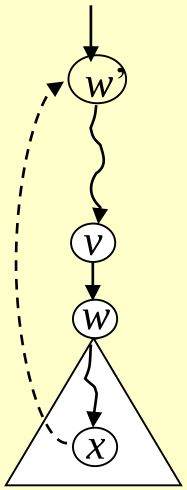
Any relation between Discovery time and articulation point ?



Assume that $(a,b) \Leftrightarrow a \rightarrow b$

Tree edge : (a,b) $a < b$

Back edge : (a,b) $a > b$



If there is a back edge from x to a proper ancestor of v, then v is reachable from x.

Finding Articulation Points

- A DFS tree can be used to discover articulation points in $\Theta(n + m)$ time.
  - We start with a program that computes a DFS tree labeling the vertices with their discovery times.
  - We also compute a function called low(v) that can be used to characterize each vertex as an articulation or non-articulation point.
  - The root of the DFS tree will be treated as a special case:
    - The root has a $d[]$ value of 1.

Finding Articulation Points

- The root of the DFS tree is an articulation point if and only if it has two or more children.
  - Suppose the root has two or more children.
    - Recall that back edges never link vertices between two different subtrees.
    - So, the subtrees are only linked through the root vertex and its removal will cause two or more connected components (i.e. the root is an articulation point).
  - Suppose the root is an articulation point.
    - This means that its removal would produce two or more connected components each previously connected to this root vertex.
    - So, the root has two or more children.

Definition of low(v)

– Definition. The value of *low*(v) is the discovery time of the vertex closest to the root and reachable from v by following zero or more tree edges downward, and then at most one back edge.

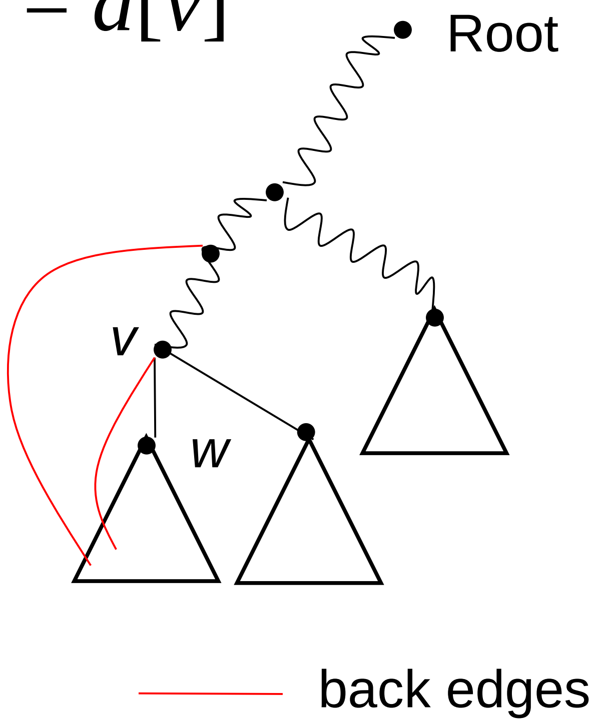
– We can efficiently compute Low by performing a postorder traversal of the depth-first spanning tree.

low[v] = min{
d[*v*],
lowest d[w] among all back edges (v,w)
lowest low[w] among all tree edges (v,w)
}

– In English: low(v) < *d*[*v*] indicates if there is another way to reach v which is not via its parent

Low(v)

- Observe that if there is a back edge from somewhere below v to above v in the tree, then low(v) < $d[v]$
- Otherwise low(v) = $d[v]$



Finding Articulation Points

- Let v be a non-root vertex of the DFS tree T.
- Then v is an articulation point of G if and only if there is a child w of v with $low(w) >= d[v]$.

Articulation Points: Pseudocode

```
Data: color[V], time, prev[V],d[V], f[V], low[V]
```

```
DFS(G) // where prog starts
{
    for each vertex u ∈ V
    {
        color[u] = WHITE;
         prev[u]=NIL;
          low[u]=inf;
         f[u]=inf; d[u]=inf;
    }
    time = 0;
    for each vertex u ∈ V
      if (color[u] == WHITE)
          DFS_Visit(u);
}
```

Articulation Points: Pseudocode

```
DFS_Visit(v)
{ color[v]=GREY;time=time+1;d[v] = time;
  low[v]= d[v];
  for each w ∈ Adj[v]{
    if(color[w] == WHITE){
      prev[w]=u;
       DFS_Visit(w);
       if low[w] >= d[v]
            record that vertex v is an articulation
       if (low[w] < low[v]) low[v] := low[w];
    }
    else if w is not the parent of v then
          //--- (v,w) is a BACK edge
 `         if (d[w] < low[v]) low[v] := d[w];
  }
  color[v] = BLACK;  time = time+1;   f[v] = time;
}
```

Special Case

- When “v” is a root of the DFS tree, you have to check it manually.

Source

- Mark Allen Weiss – Data Structure and Algorithm Analysis in C
  - Articulation Point
- Exercise:
  - Cormen – Exercise 22-2
  - What is bridge? How can it be detected?