Linear Programming

Introduction to Linear Programming

- A Linear Programming model seeks to maximize or minimize a linear function, subject to a set of linear constraints.
- The linear model consists of the following components:
  - A set of decision variables.
  - An objective function.
  - A set of constraints.

Problem

- Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains.
- A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by $14.
- A train sells for $20 and uses $9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by 10$.
- The manufacture of wooden solders and trains requires two types of skilled labor: carpentry and finishing .
- A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor.
- A train requires 1 hours of finishing labor and 1 hours of carpentry labor.
- Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week.
- Giapetto wants to maximize weekly profit (revenues – costs).
- Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.

Decision Variable

- Decision variables should completely describe the decisions to be made.
- In this problem, it is required to know how many soldiers and trains should be manufactured each week.
  - x_1 = number of soldiers produced each week
  - x_2 = number of trains produced each week.

Objective Functions

- Decision maker wants to maximize or minimize some function of the decision variables. This function is called objective function.
- In this problem it's required to maximize
  - weekly revenues – (raw materials purchase cost) – (other variable costs)
  - where,
    - Weekly revenues

      = weekly_revenues_from_soldiers + weekly_revenues_from_trains

      $=27x_1 + 21x_2$
    - Weekly raw material cost = $10x_1+9x_2$
    - Other weekly variable costs = $14x_1 + 10x_2$
  - So maximize $(27x_1 + 21x_2)-(10x_1+9x_2)-(14x_1 + 10x_2)= 3x_1+2x_2$
  - So Objective function is
    - **Maximize $z= 3x_1+2x_2$**

Constraints:

- Each week, no more than 100 hours of finishing time may be used.
  - $2x_1+x_2\leq 100$
- Each week, no more than 80 hours of carpentry time may be used.
  - $x_1+x_2\leq 80$
- Because of limited demand, at most 40 soldiers should be produced each week.
  - $x_1\leq 40$

Sign Restrictions

- $x_1 \geq 0$
- $x_2 \geq 0$

Optimized Model:

- Maximize $z = 3x_1 + 2x_2$
- Subject to
  - $2x_1 + x_2 \leq 100$
  - $x_1 + x_2 \leq 80$
  - $x_1 \leq 40$
  - $x_1 \geq 0$
  - $x_2 \geq 0$

Introduction to Linear Programming

- The Importance of Linear Programming
  - Many real world problems lend themselves to linear programming modeling.
  - Many real world problems can be approximated by linear models.
  - There are well-known successful applications in:
    - Manufacturing
    - Marketing
    - Finance (investment)
    - Advertising
    - Agriculture

Introduction to Linear Programming

- Assumptions of the linear programming model
  - The parameter values are known with ***certainty.***
  - There are ***no interactions*** between the decision variables (the additivity assumption).
  - The ***Continuity*** assumption: Variables can take on any value within a given feasible range.

Definitions

- Decision Variable
- Objective Function
- Sign Restrictions
  - Can the decision variable only assume nonnegative values or allowed to assume both positive and negative values.

Definitions

- **Linear Function:**
  - A function $f(x_1,x_2..x_n)$ of $x_1,x_2…x_n$ is a linear function if and only if for some constants c_1, $c_2…c_n$, $f(x_1,x_2..x_n) = c_1x_1+c_2x_2+..+c_nx_n$
  - Example: $f(x_1,x_2) = 2x_1+x_2$ is a linear function
- **Feasible Region:**
  - set of all points satisfying all the LP’s constraints and all the LP’s sign restrictions.

Types of Mathematical Programming

- **Linear Programs (LP):** the objective and constraint functions are linear and the decision variables are continuous.
- **Integer Linear Programs (ILP):** one or more of the decision variables are restricted to integer values only and the functions are linear.
  - Pure IP: all decision variables are integer.
  - Mixed IP (MIP): some decision variables are integer, others are continuous.
  - 1/0 MIP: some or all decision variables are further restricted to be valued either “1” or “0”.
- **Nonlinear Programs:** one or more of the functions is not linear.

Linear Programming

- General symbolic form

Maximize: $c_1 x_1 + c_2 x_2 + \cdots c_n x_n$ } Objective

Subject to:

$$\left.\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \{\le, \ge, =\} \;\; b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \{\le, \ge, =\} \;\; b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \{\le, \ge, =\} \;\; b_m \end{array}\right\} \text{Constraints}$$

$0 \le x_j, \quad j = 1, \ldots, n$ } Bounds

…where a_{ij}, b_i, and c_i are the model **parameters**.

Solving Mathematical Programming Problems

- Graphical method
  - Only useful for 2 decision variables (maybe 3 if you can handle 3-D chart)
- Simplex method
  - Efficient algorithm to solve LP problems by performing matrix operations on the LP Tableau
  - Developed by George Dantzig (1947)
  - Can be used to solve small LP problems by hand
- Software packages
  - MS Excel
  - Lindo, LPSolve
  - AMPL/CPLEX: modeling language and “solver” for large and complex LP/IP problems
- Sub-Optimal Algorithms (Heuristics)
  - Simulated annealing
  - Genetic algorithms
  - Tabu search

The Galaxy Industries Production Problem – A Prototype Example

- Galaxy manufactures two toy doll models:
  - Space Ray.
  - Zapper.
- Resources are limited to
  - 1000 pounds of special plastic.
  - 40 hours of production time per week.

The Galaxy Industries Production Problem – A Prototype Example

- Marketing requirement
  - Total production cannot exceed 700 dozens.
  - Number of dozens of Space Rays cannot exceed number of dozens of Zappers by more than 350.

- Technological input
  - Space Rays requires 2 pounds of plastic and 3 minutes of labor per dozen.
  - Zappers requires 1 pound of plastic and 4 minutes of labor per dozen.

The Galaxy Industries Production Problem – A Prototype Example

- The current production plan calls for:
  - Producing as much as possible of the more profitable product, Space Ray ($8 profit per dozen).
  - Use resources left over to produce Zappers ($5 profit per dozen), while remaining within the marketing guidelines.
- The current production plan consists of:

Space Rays = 450 dozen
Zapper = 100 dozen
Profit = $4100 per week

8(450) + 5(100)

Management is seeking a production schedule that will increase the company's profit.

A linear programming model can provide an insight and an intelligent solution to this problem.

The Galaxy Linear Programming Model

- Decisions variables:
  - X_1 = Weekly production level of Space Rays (in dozens)
  - X_2 = Weekly production level of Zappers (in dozens).
- Objective Function:
  - Weekly profit, to be maximized

The Galaxy Linear Programming Model

Max $8X_1 + 5X_2$ (Weekly profit)

subject to

$2X_1 + 1X_2 \leq 1000$ (Plastic)

$3X_1 + 4X_2 \leq 2400$ (Production Time)

$X_1 + X_2 \leq 700$ (Total production)

$X_1 - X_2 \leq 350$ (Mix)

$X_j >= 0, \ j = 1,2$ (Nonnegativity)

2.3 The Graphical Analysis of Linear Programming

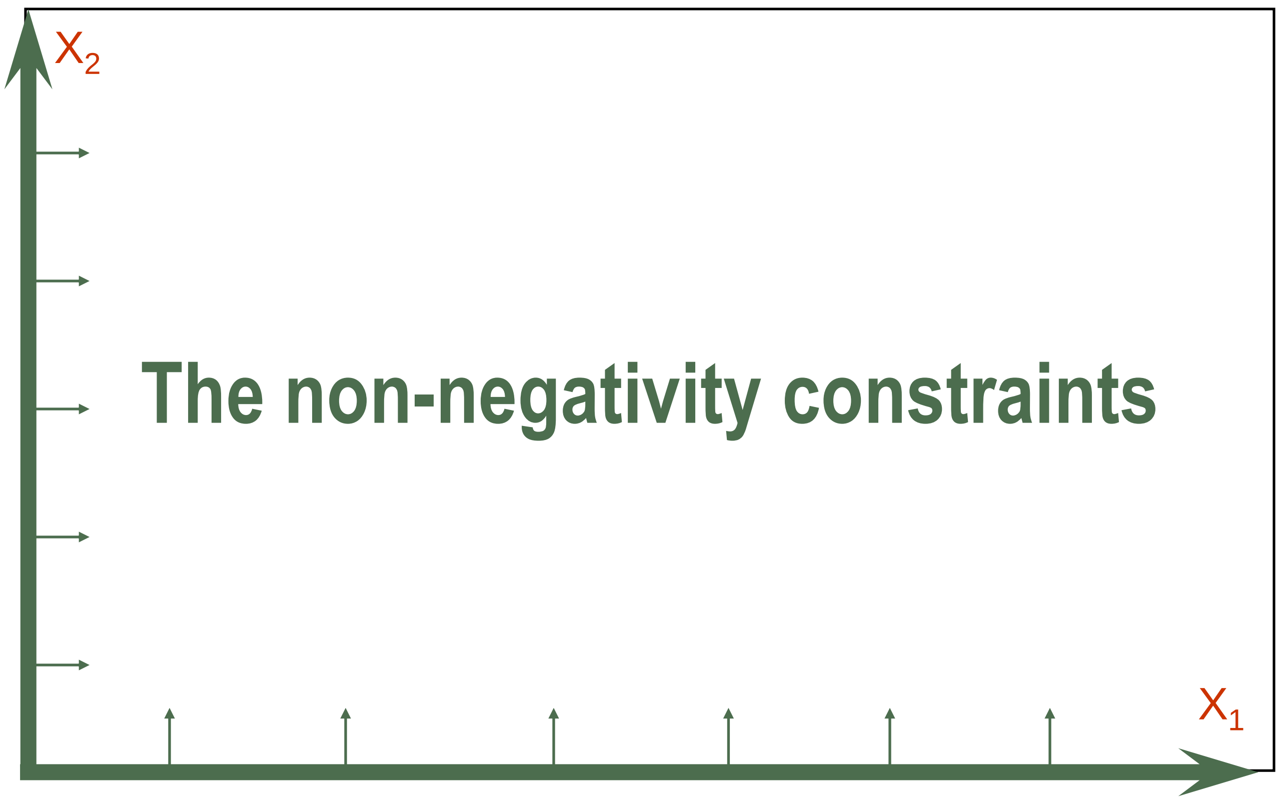
The set of all points that satisfy all the constraints of the model is called

a

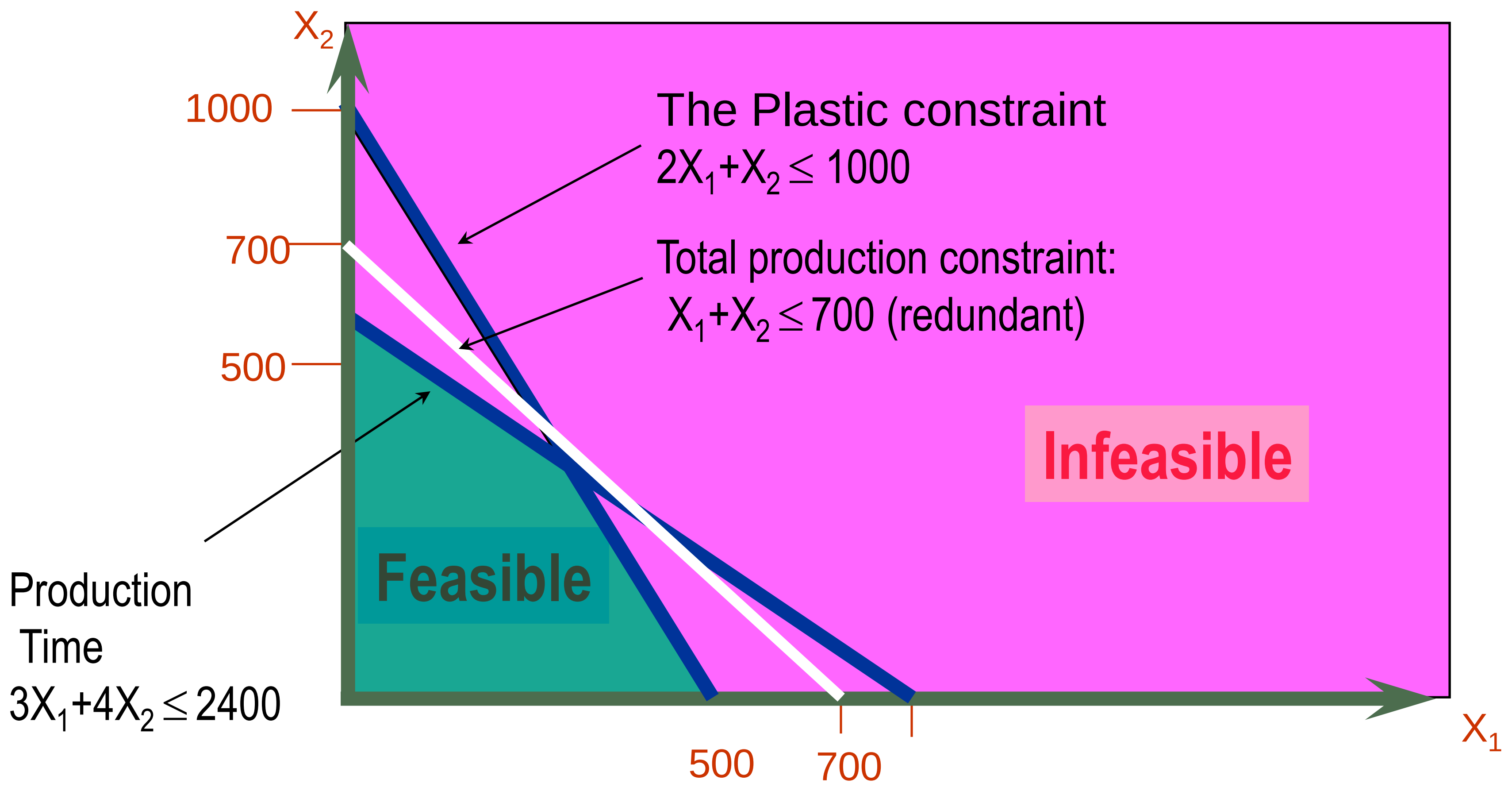
FEASIBLE REGION

Using a graphical presentation we can represent all the constraints, the objective function, and the three types of feasible points.

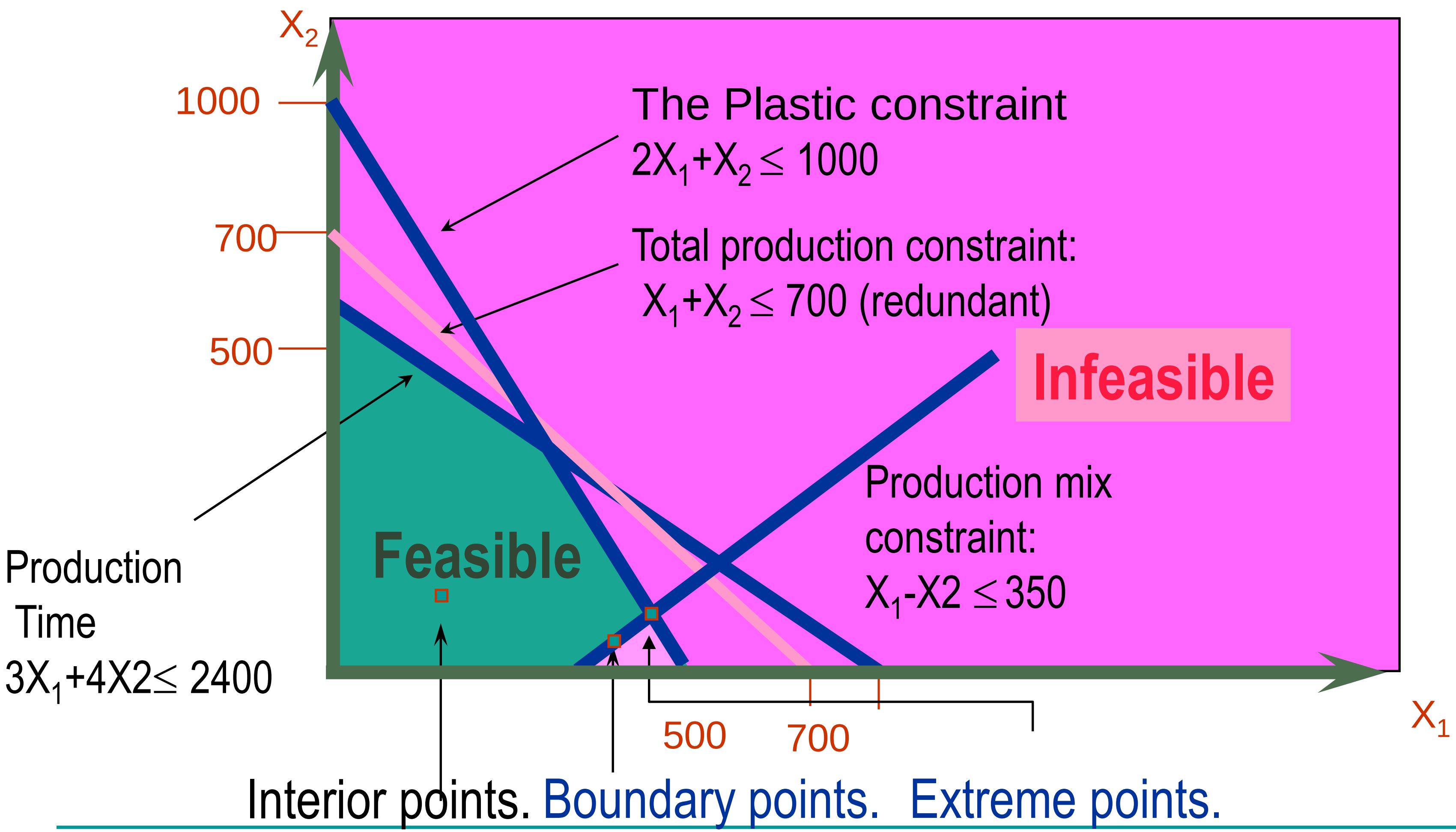
Graphical Analysis – the Feasible Region



Graphical Analysis – the Feasible Region



Graphical Analysis – the Feasible Region



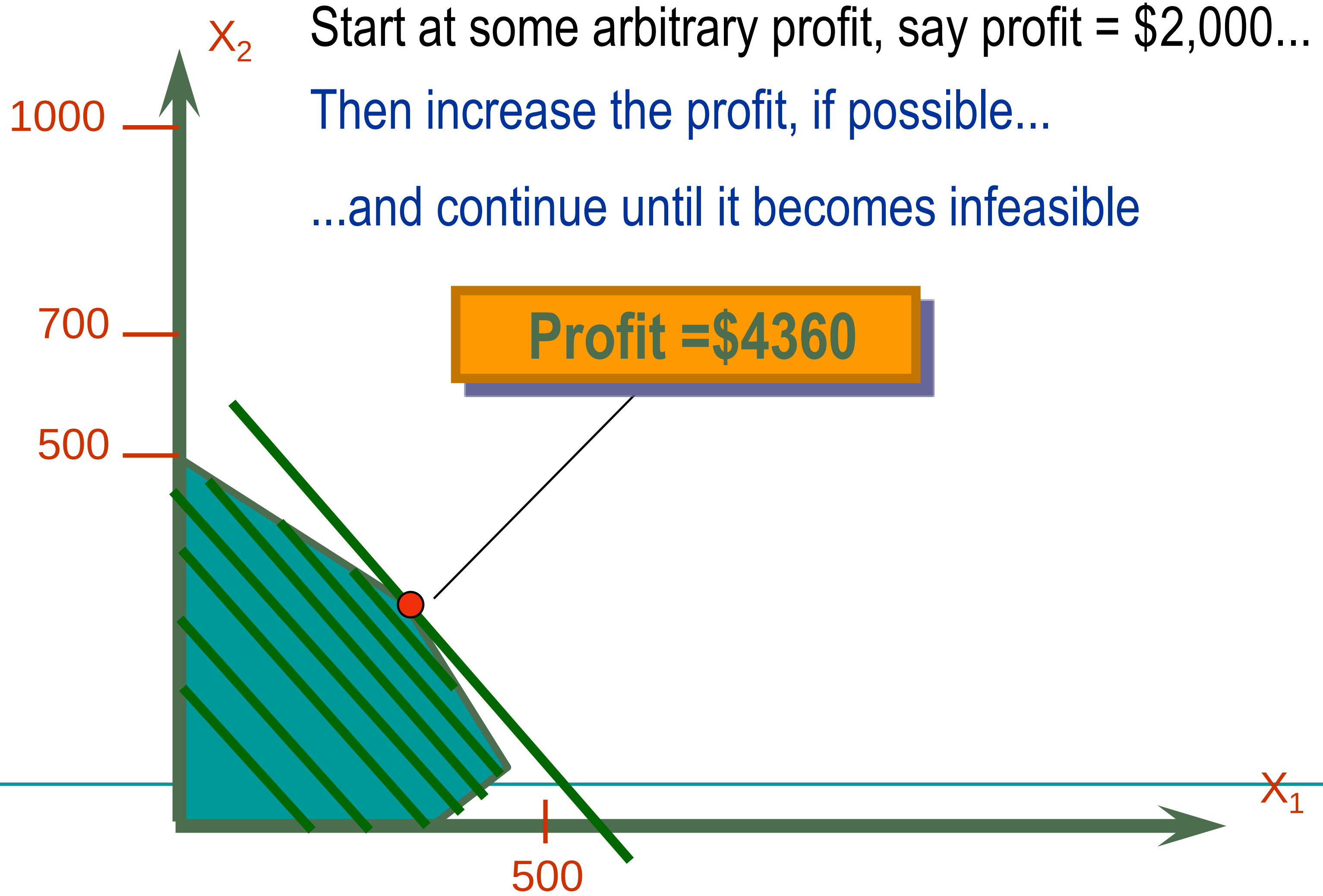
- **There are three types of feasible points**

The search for an optimal solution

Start at some arbitrary profit, say profit = $2,000...

Then increase the profit, if possible...

...and continue until it becomes infeasible



Summary of the optimal solution

Space Rays = 320 dozen

Zappers = 360 dozen

Profit = $4360

- This solution utilizes all the plastic and all the production hours.
- Total production is only 680 (not 700).
- Space Rays production exceeds Zappers production by only 40 dozens.

Another Simple Example

- Example
  - A steel company must decide how to allocate production time on a rolling mill. The mill takes unfinished slabs of steel as input and can produce either of two products: bands and coils. The products come off the mill at different rates and have different profitability:

	Tons/ hour	Profit/ ton
Bands	200	$25
Coils	140	$30

  - The weekly production that can be justified based on current and forecast orders are:

Maximum tons:	Bands	6,000
	Coils	4,000

Another Simple Example – Solution

- The question facing the company:
  - If 40 hours of production time are available, how many tons of bands and coils should be produced to bring the greatest profit?
- Constructing the Verbal model
  - Put the objective and constraints into words.
  - For constraints, use the form:
    - {a verbal description of the LHS} {a relationship} {an RHS constant}

Maximize: total profit

Subject to: total number of production hours $\leq$ 40
tons of bands produced $\leq$ 6,000
tons of coils produced $\leq$ 4,000

Another Simple Example – Solution (2)

- Define the decision variables:
  - x_B number of tons of bands produced.
  - x_C number of tons of coils produced.
- Construct the symbolic model

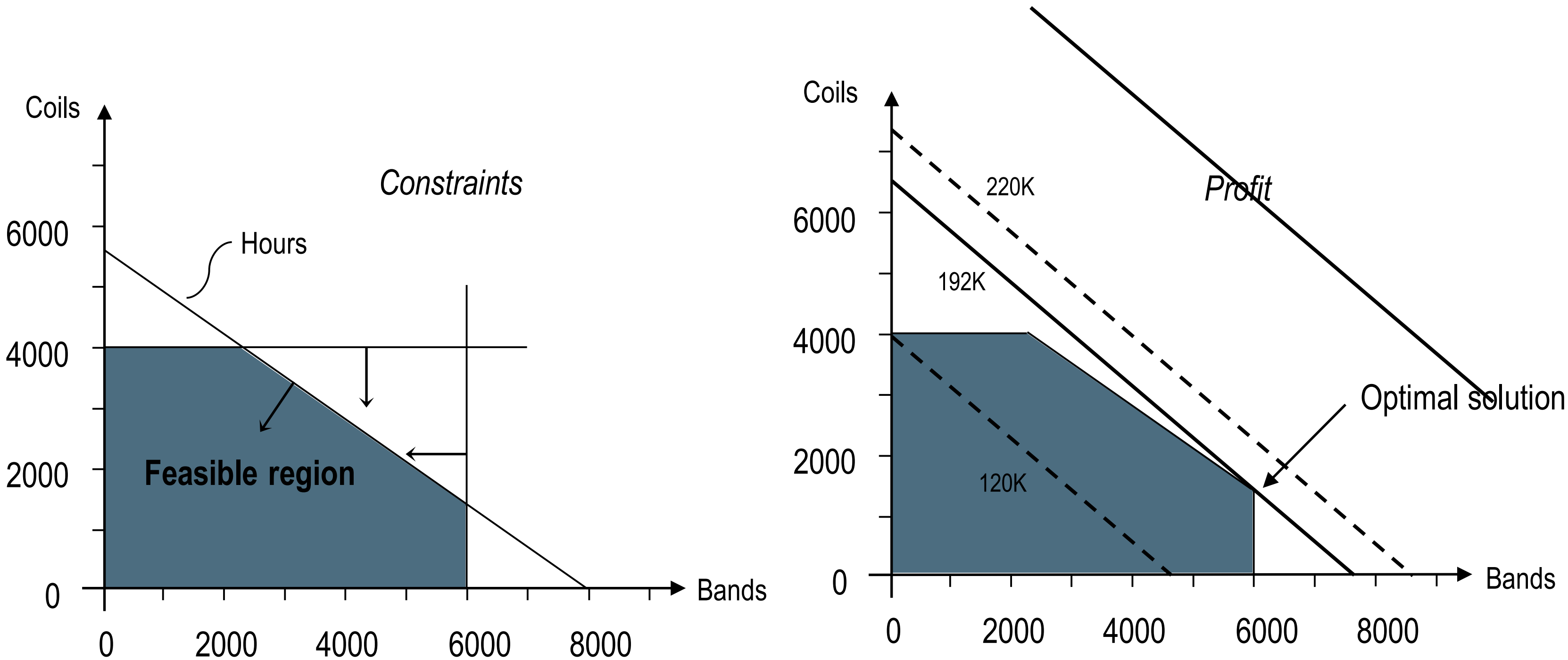
Maximize: $$25x_B + 30x_C$$

Subject to: $$(1/200)\,x_B + (1/140)\,x_C \le 40$$

$$0 \le x_B \le 6000$$

$$0 \le x_C \le 4000$$

Another Simple Example – Solution (3)



Graphical Solution – Wyndor Glass Problem

$$\max \quad Z = 3x_1 + 5x_2$$

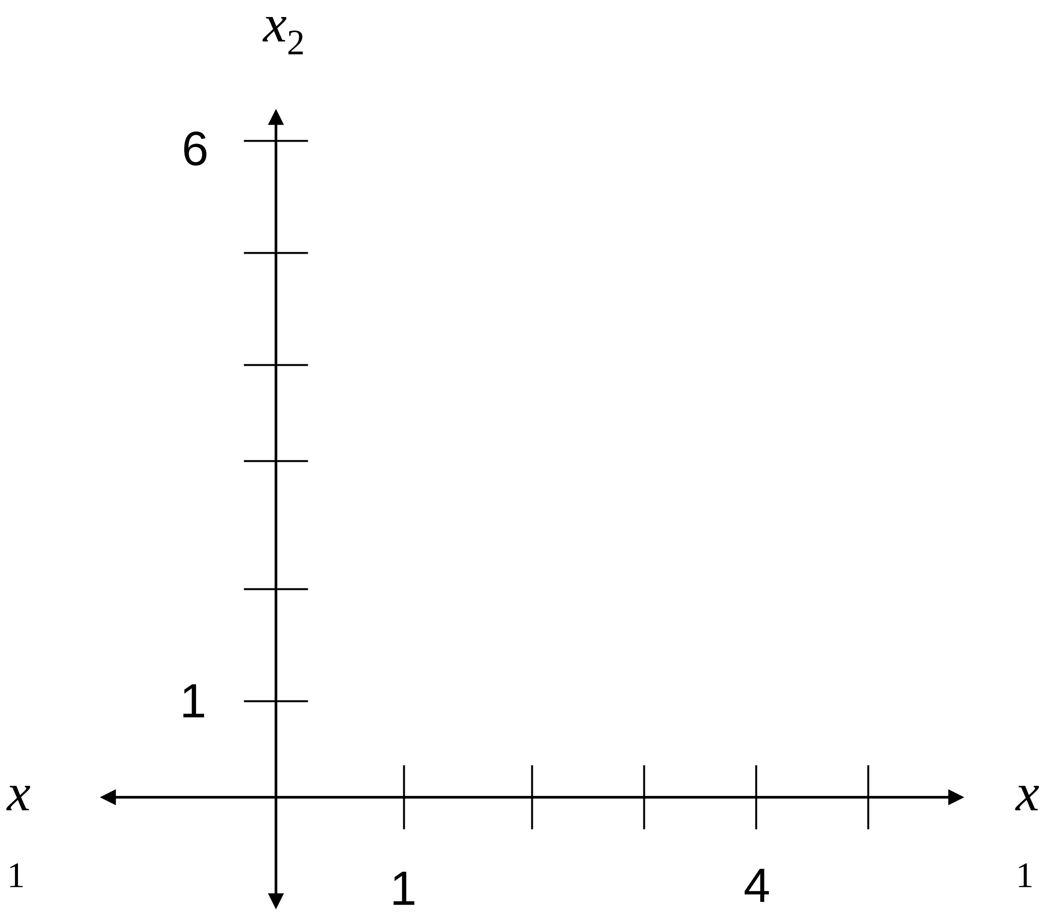
$$s.t.$$

$$1x_1 + 0x_2 \le 4$$

$$0x_1 + 2x_2 \le 12$$

$$3x_1 + 2x_2 \le 18$$

$$x_1 \ge 0, \quad x_2 \ge 0.$$



What is the feasible region?

Is the feasible region convex?

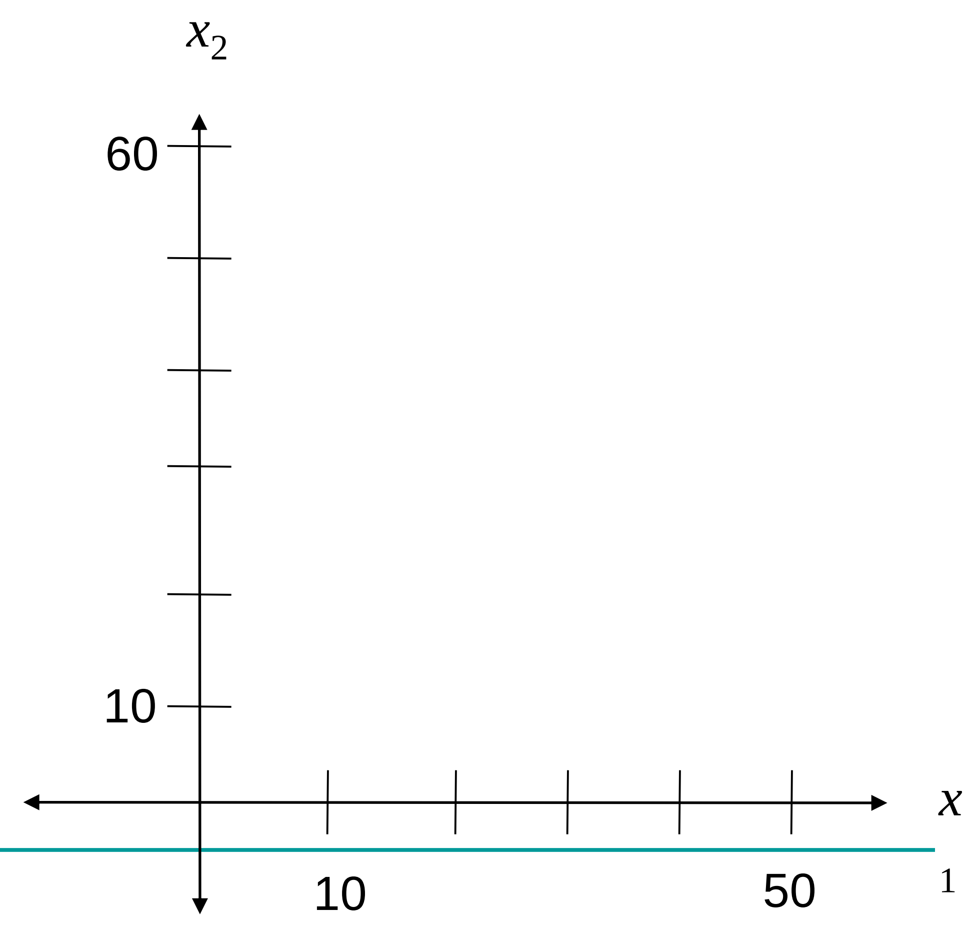
Graphical Solution Method – cont.

- **Minimization Problem –** objective function moves in a direction that reduces the objective value.

Graphical Solution Method – cont.

- **Multiple Optimal Solutions -** the objective function is parallel to a constraint as it leaves the feasible region.

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1/40 + x_2/60 \leq 1 \\ & x_1/50 + x_2/50 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

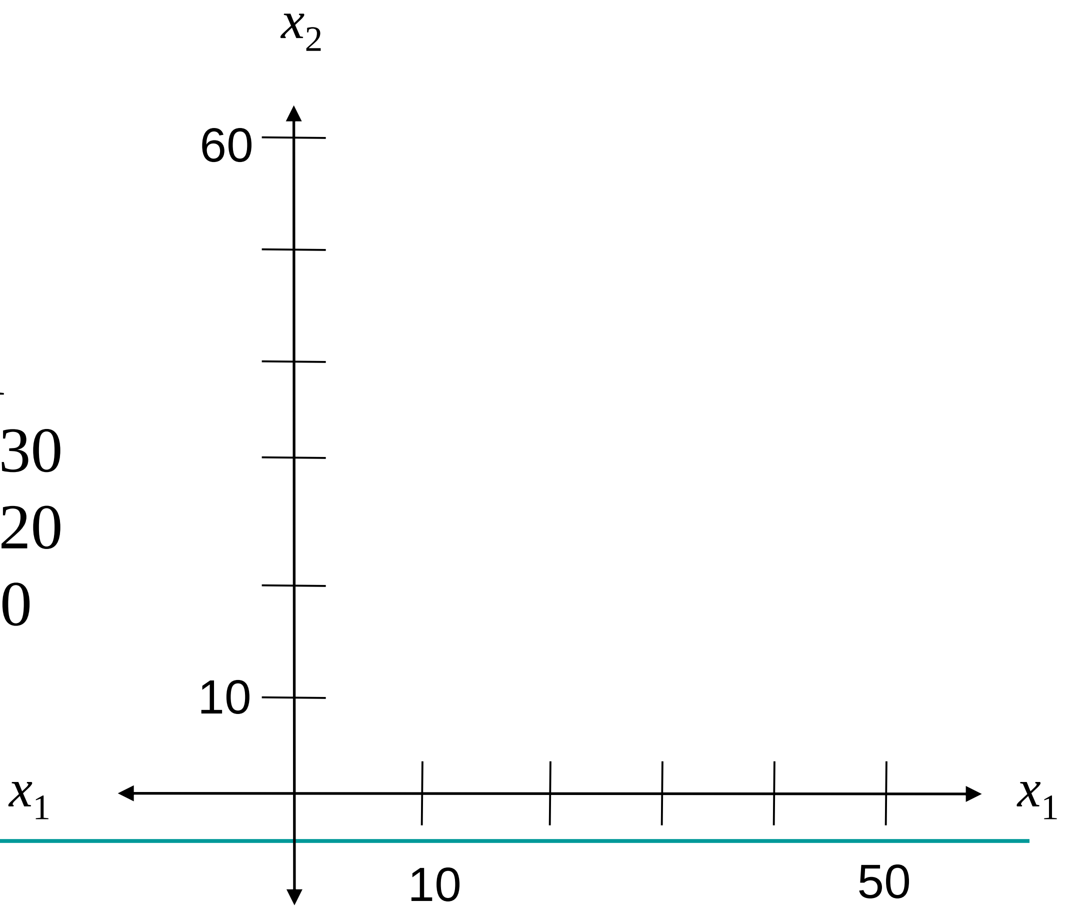


Can you have exactly two optimal solutions?

Graphical Solution Method – cont.

- **Infeasible LP –** the feasible region is empty

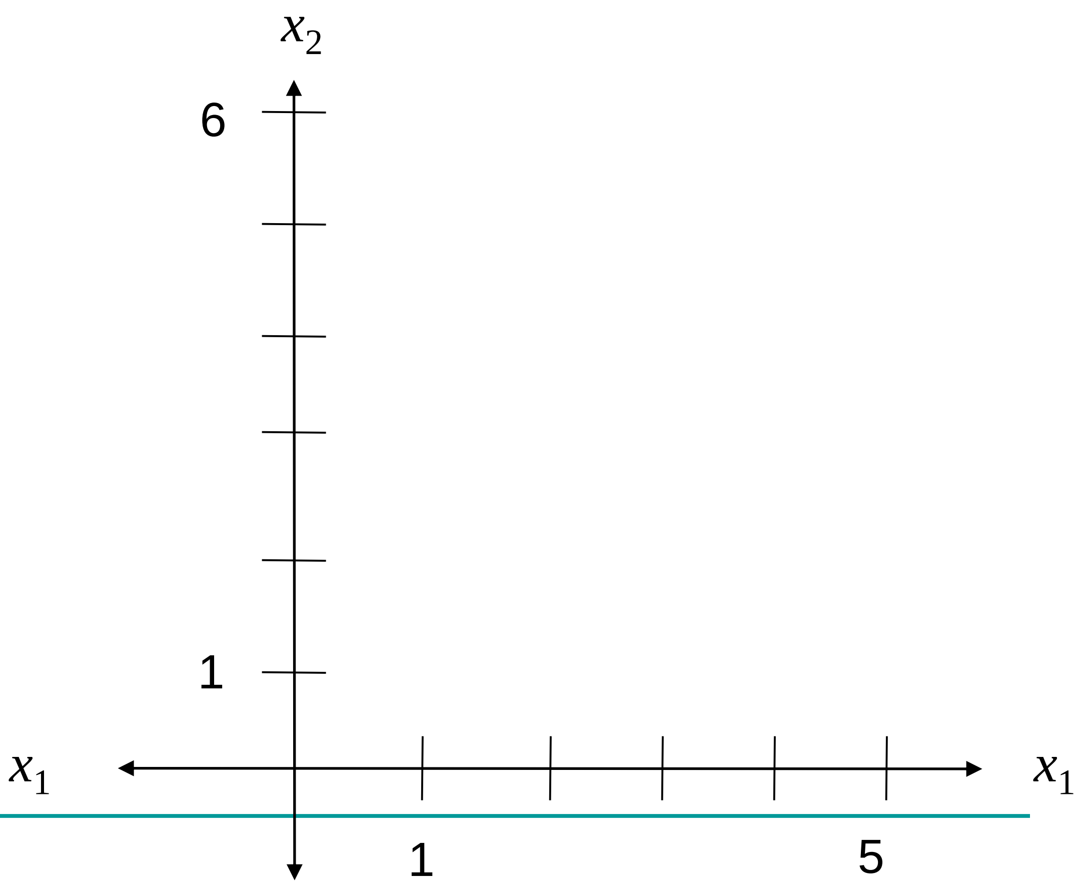
$$
\begin{aligned}
\max\ & 3x_1 + 2x_2 \\
\text{s.t.}\ & x_1/40 + x_2/60 \le 1 \\
& x_1/50 + x_2/50 \le 1 \\
& x_1 \ge 30 \\
& x_2 \ge 20 \\
& x_1 , x_2 \ge 0
\end{aligned}
$$



Graphical Solution Method – cont.

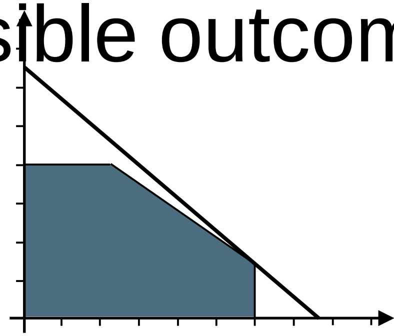
- **Unbounded LP –** the feasible region is unbounded, goes to infinity

$$\begin{aligned} \max \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leq 1 \\ & 2x_1 + x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

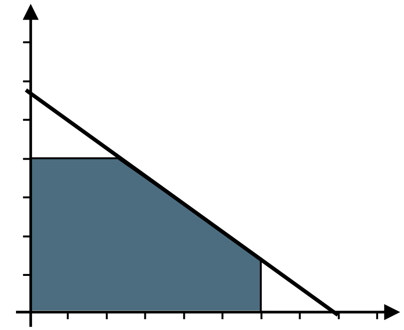


Solving LP Problems Graphically – Outcomes

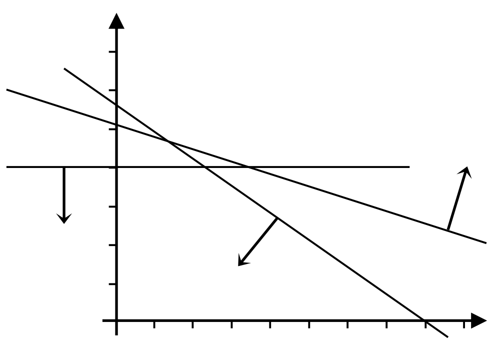
- 4 possible outcomes:



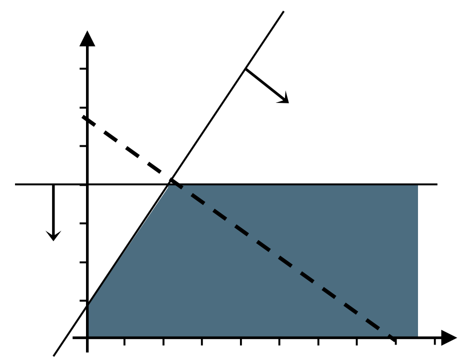
Unique Optimal Solution



Alternate Optimal Solutions



No Feasible Solution



Unbounded Optimal Solution

In-Class Problem #1

- A diet is being prepared for the UoA's Lister Hall. You need to feed the students at the least cost, but the diet must have between 1800 and 3600 calories. No more than 1400 calories can be from starch, no fewer than 400 can be protein, and no more than 150 can be fat. The diet is to be made of two foods: A and B. Food A costs $1.75 per pound and contains 600 cal/lb, 400 of which is from protein and 200 from starch. No more than 2 pounds of food A can be used per resident. Food B costs $2.50 per pound and contains 900 cal/lb, 700 from starch, 100 from protein, and 100 from fat.

- How much of each food should be in the diet?

In-Class Problem #2

- You want to mix two fuels, A and B, to run your truck fleet at minimum cost. Your fleet requires at least 3000 litres per month, you have storage capacity for 4000 litres of mixed fuel, and your supplier currently has 2000 litres of fuel type A and 4000 litres of fuel type B available. Fuel A costs $1.20 per litre and has an octane of 90, while fuel B costs $0.90 per litre and has an octane of 75. Your trucks need an octane of at least 80, and the octane level of the mixture is a simple weighted average (by volume) of the input fuels.

- How much of each type of fuel should you buy for your truck fleet mixture?

In-Class Problem #3

- Your facility can complete jobs on your own assembly machine, A, or another machine, B, in the fabrication shop across the street. Each of your jobs can be finished on either machine, but it costs $2 per job on machine A and $4 per job on machine B. You’re already committed to at least 10 jobs per week on the machine across the street, and need to find work for at least 3 personnel you have on extended contracts (40 hours per week each). It takes 4 hours to do each job on machine A and 6 hours to do it on machine B (you provide the personnel for machine B as well).
- How many jobs should you assign to each machine?
  - Don’t worry about integrality… the solution comes out integer anyway. ☺

In-Class Problem #4

- Solve the following linear programming problem graphically.
  - Objective Function: $\min: \; 4A + 5B + 6C$
  - Subject to:

$$A + B \geq 11$$

$$A - B \leq 5$$

$$-A - B + C = 0$$

$$7A + 12B \geq 35$$

$$A, B, C \geq 0$$

Preview of the Simplex Method

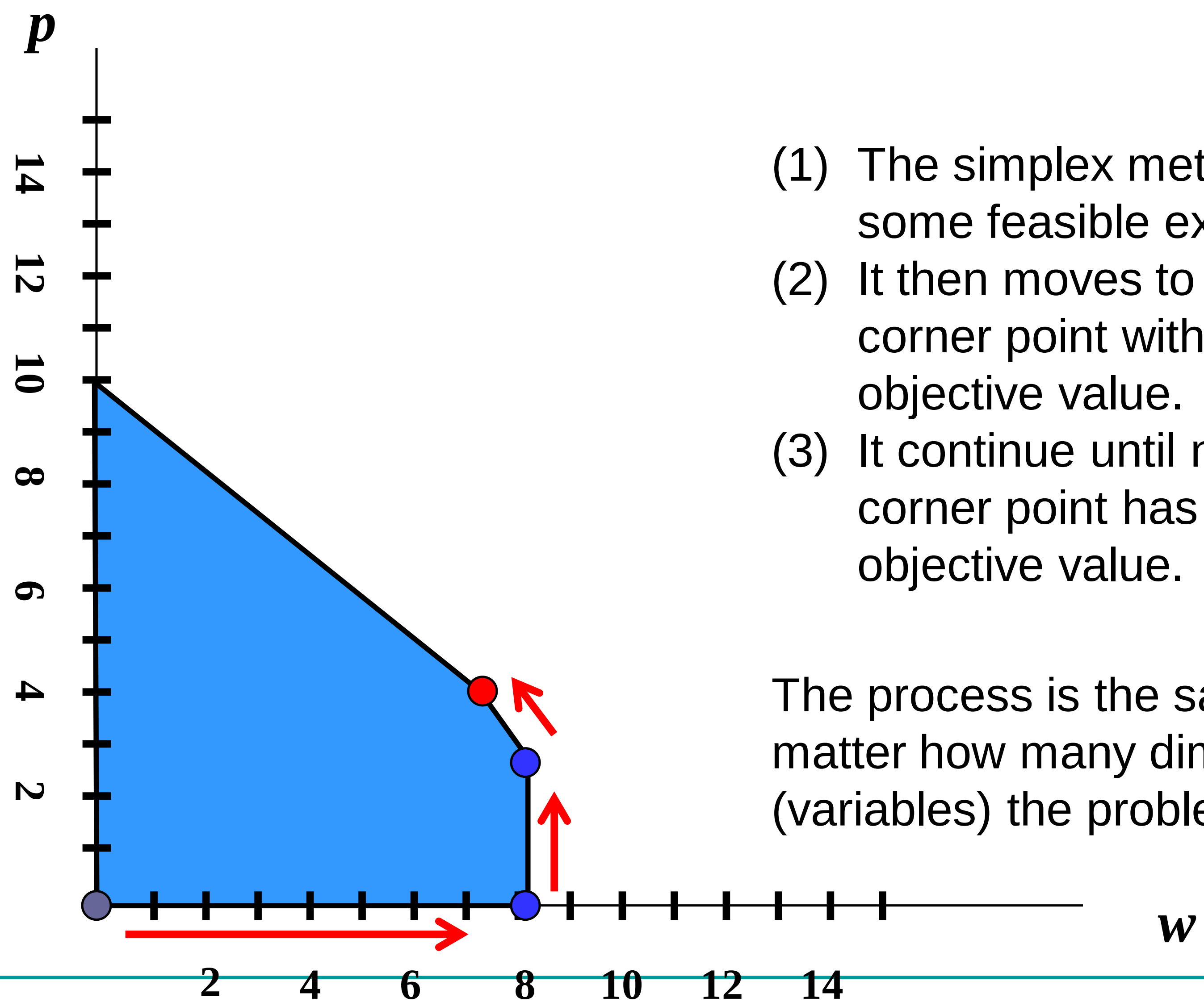
- The graphical method can be used to solve a 2-dimensional LP problem (i.e., one with 2 variables).

- An *n*-dimensional problem where $n > 2$, however, is impossible to solve graphically (well, maybe $n > 3$).

- We can use the simplex method to find solutions to these larger problems.
  - The simplex method finds the best solution by a neighbourhood search technique, essentially moving between adjacent corner points within the feasible region (corner points are adjacent if they have one binding constraint in common).

Adapted from: James Orlin, MIT, 2003

Preview of the Simplex Method (2)



(1) The simplex method starts at some feasible extreme point.
(2) It then moves to an adjacent corner point with better objective value.
(3) It continue until no adjacent corner point has a better objective value.

The process is the same no matter how many dimensions (variables) the problem has.

Adapted from: James Orlin, MIT, 2003