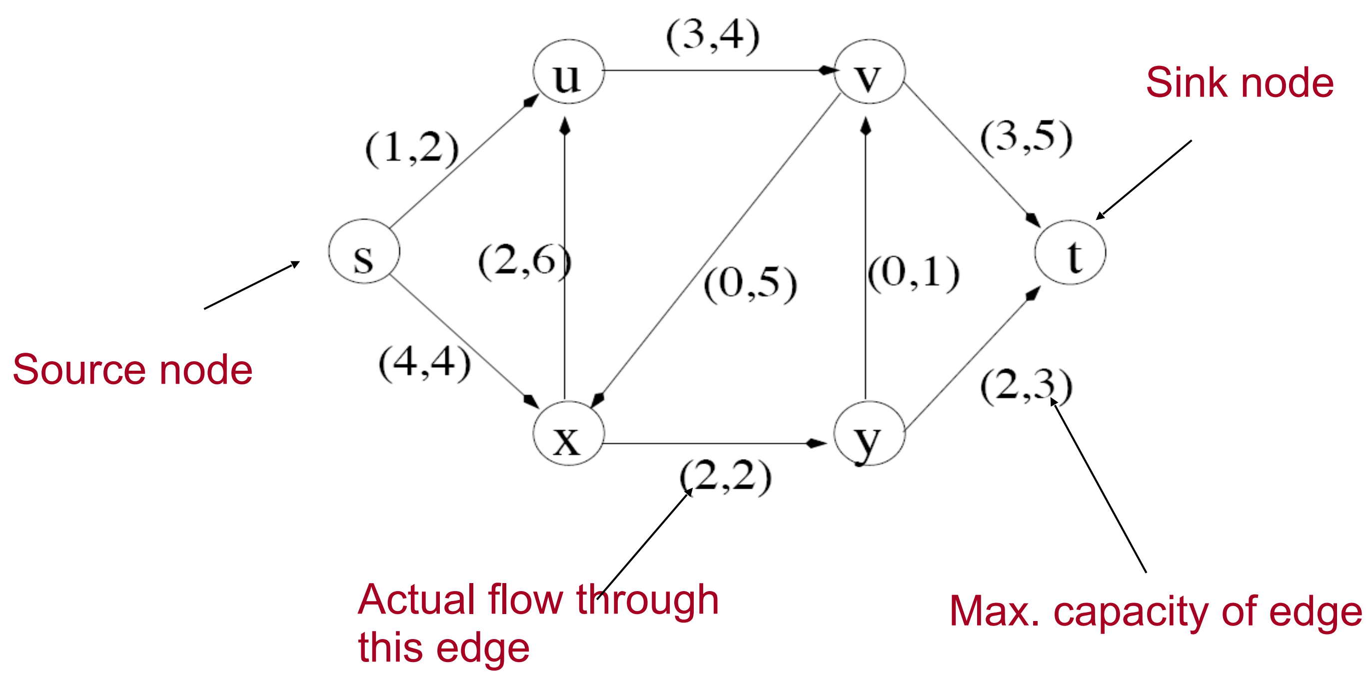
# NETWORK FLOW

# Usage Places

| Network | Vertex | Edges | Flow |
| --- | --- | --- | --- |
| communication | telephone exchanges computers, satellites | cables, fiber optics, microwave relays | voice, video, packets |
| circuits | gates, registers, processors | wires | current |
| mechanical | Joints | rods, beams, springs | heat, energy |
| hydraulic | reservoirs, pumping stations, lakes | pipelines | fluid, oil |
| financial | stocks, currency | transactions | money |
| transportation | airports, rail yards, street intersections | highways, railways, airway routes | freight, vehicles, passengers |
| chemical | sites | bonds | energy |

# Example Graph



# Simplified Model

The network is modeled simply as

a) a directed graph G = (V,E) with
b) non-negative capacity on each edge,
c) a single source node, *s*, and
d) a single sink node, *t*

# Some assumptions…

We will simplify our discussion by assuming the following:

(i) No edge enters *s*, the source

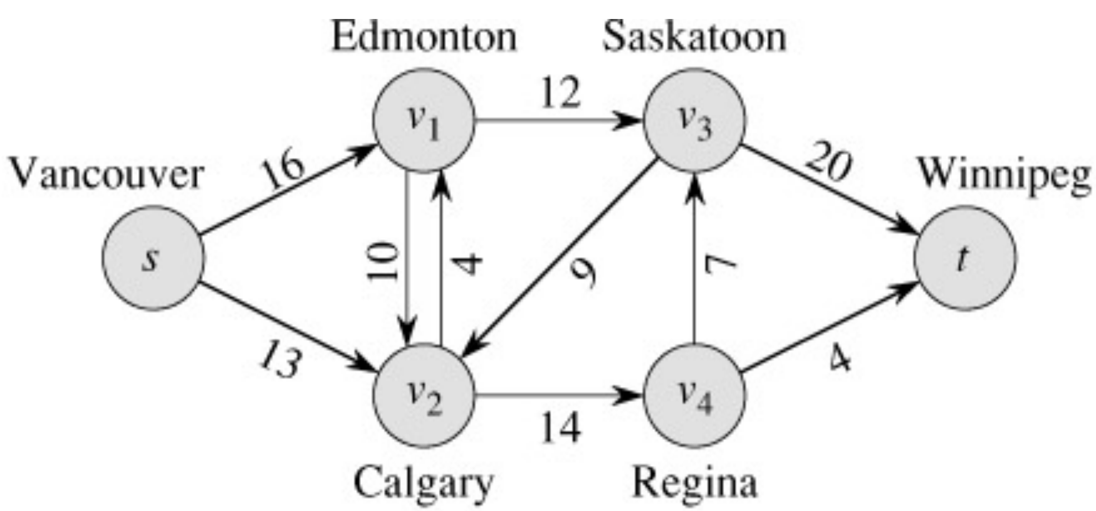
(ii) No edge leaves *t*, the sink

(iii) At least one edge is incident to each node

(iv) All capacities are integers

# What is Network Flow?

- Each edge (u,v) has a nonnegative capacity c(u,v).
- If (u,v) is not in E, assume c(u,v)=0.
- We have a source s, and a sink t.
- Assume that every vertex v in V is on some path from s to t.
- $c(s,v_1)=16$; $c(v_1,s)=0$; $c(v_2,v_3)=0$



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- *Flow conservation:* $\forall u \in V - \{s, t\}$:

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} f(v,u)$$

# Flow in a Flow Network

– A flow in the network is an integer-valued function f defined on the edges of G satisfying $0 \le f(i,j) \le c(i,j)$ for every edge (i,j) in E.

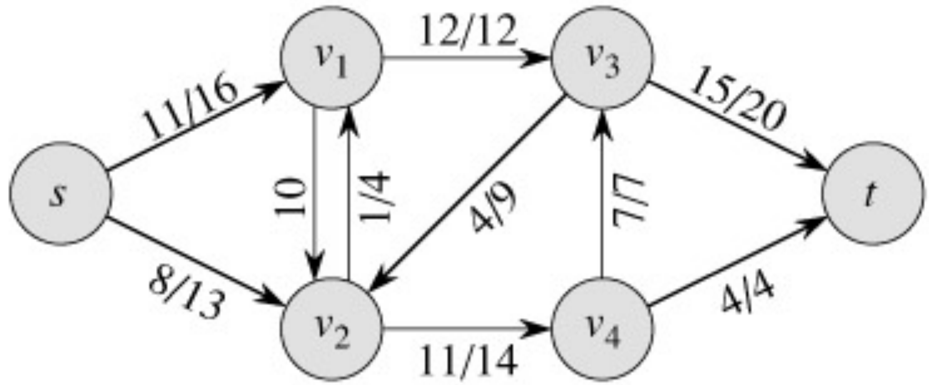
# The Value of a Flow

- The value of a flow is given by

$$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t)$$

- This it the total flow leaving $s$ = the total flow arriving in $t$

# Example:



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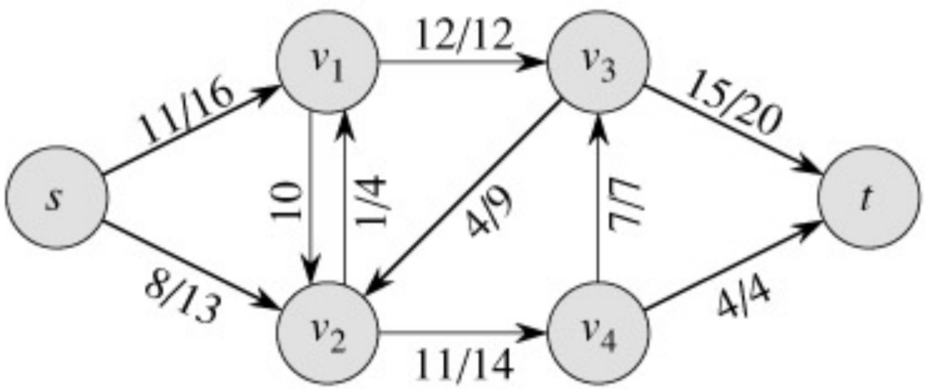


$|f| = f(s, v_1) + f(s, v_2) + f(s, v_3) + f(s, v_4) + f(s, t) =$

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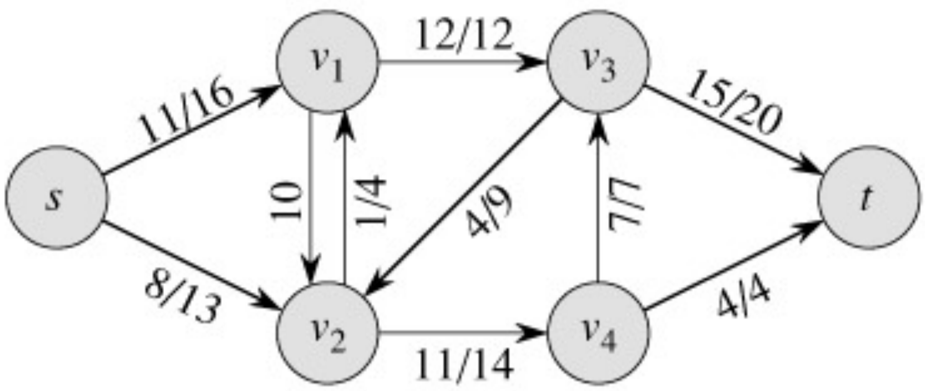


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$11 + 8 + 0 + 0 + 0 = 19$

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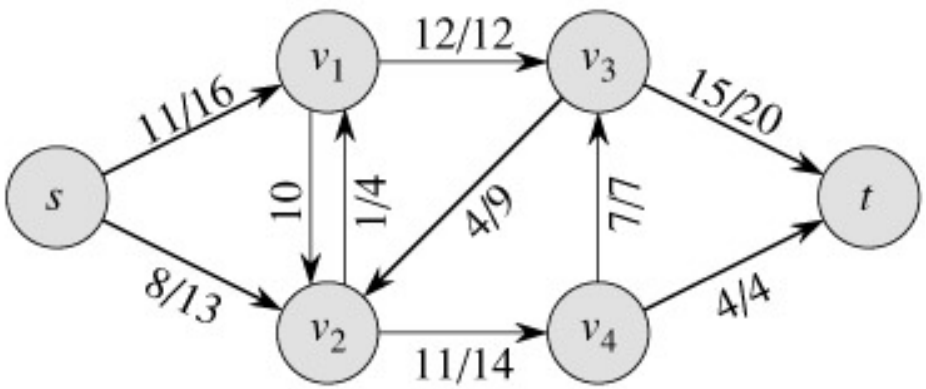


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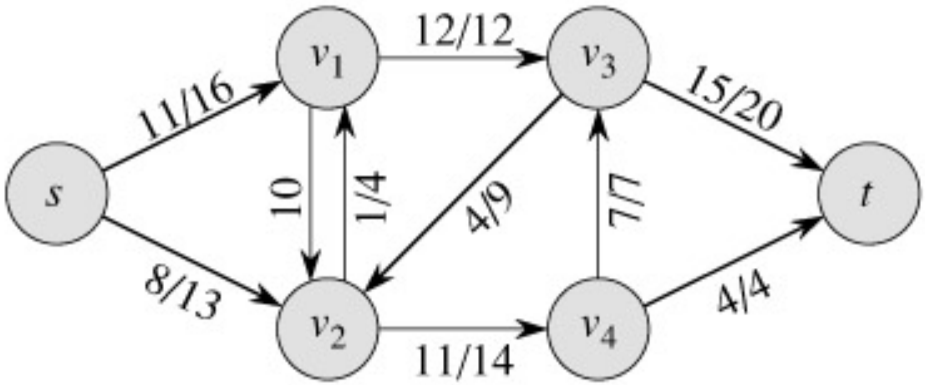


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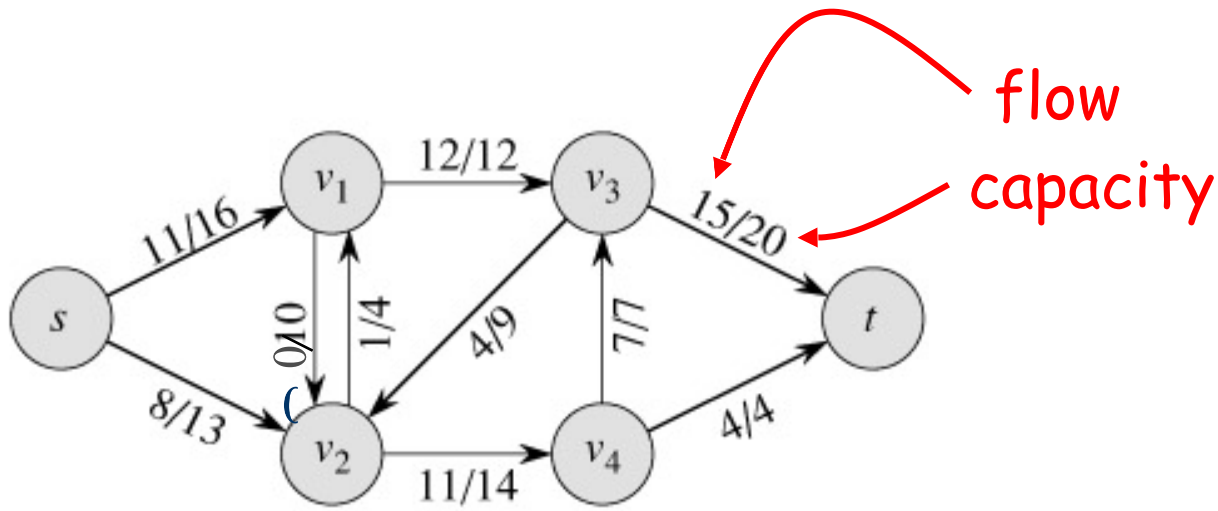
$|f| = f(s, t) + f(v_1, t) + f(v_2, t) + f(v_3, t) + f(v_4, t) =$

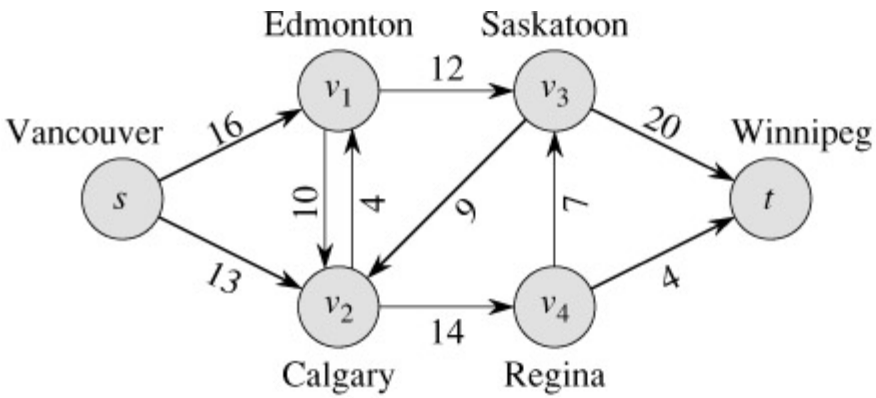
$0 + 0 + 0 + 15 + 4 = 19$

# Example of a Flow




- $f(v_2, v_1) = 1$, $c(v_2, v_1) = 4$.
- $f(v_1, v_2) = -1$, $c(v_1, v_2) = 10$.
- f(v3, s) + f(v3, v1) + f(v3, v2) + f(v3, v4) + f(v3, t) =
- 0 + (-12) + 4 + (-7) + 15 = 0

# A flow in a network



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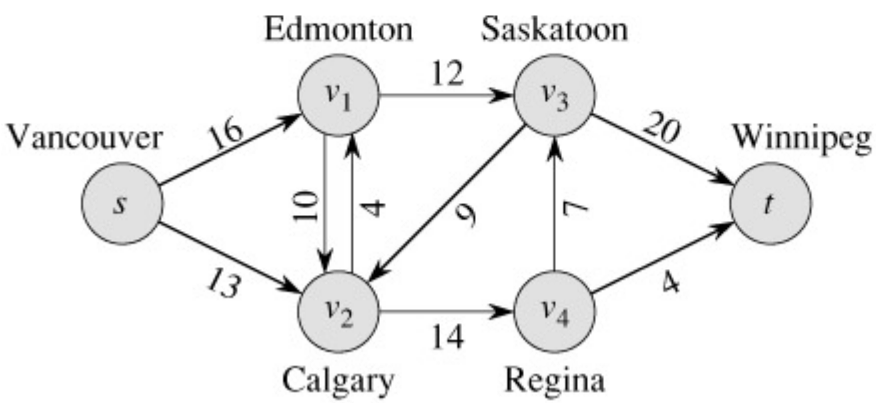
- We assume that there is only flow in one direction at a time.



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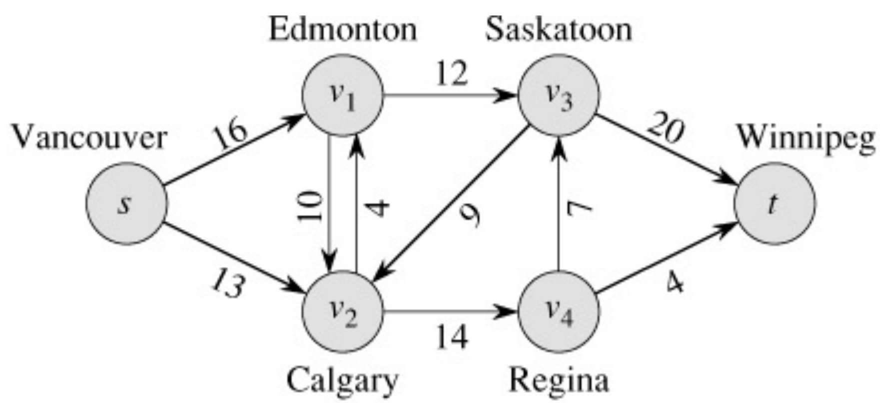


- Sending 7 trucks from Edmonton to Calgary and 3 trucks from Calgary to Edmonton has the same net effect as sending 4 trucks from Edmonton to Calgary.

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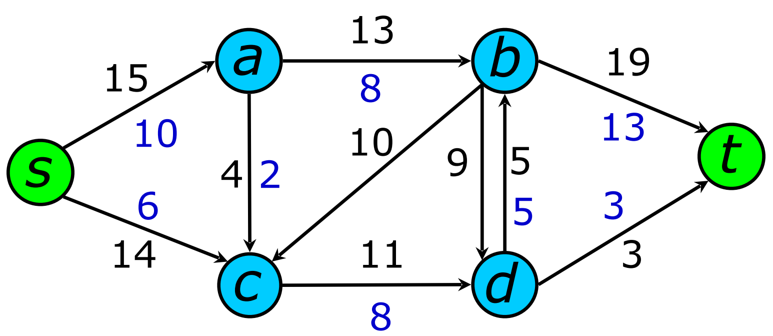
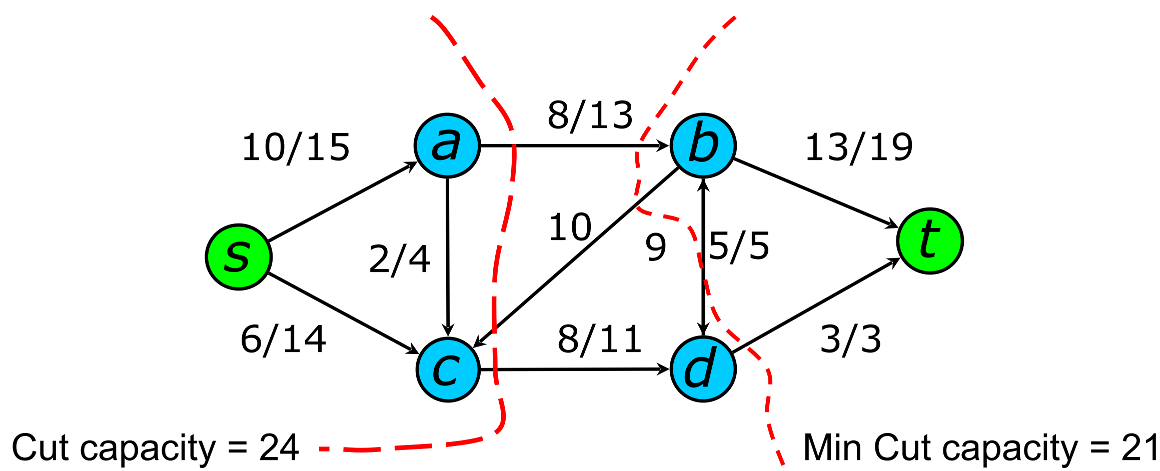


# Some Lemmas:

- Prove that, $f(s,V)=f(V,t)$
- [Lemma26.2] Prove that, $|f+f'|=|f|+|f'|$
- *Lemma: 26.3*
- *Lemma 26.4* $|f'| = |f| + |f_p| \geq |f|$
- *Lemma: 26.5* $|f| = f(S,T)$
- *Lemma: 26.6* $|f| \leq c(S,T)$

# Cuts

- A **cut** is a partition of V into S and T = V – S, such that $s \in$ S and $t \in$ T
- The **net flow** (f(S,T)) through the cut is the sum of flows f(u,v), where $s \in$ S and $t \in$ T
  - Includes negative flows back from T to S
- The **capacity** (c(S,T)) of the cut is the sum of capacities c(u,v), where $s \in$ S and $t \in$ T
  - The sum of positive capacities
- **Minimum cut** – a cut with the smallest capacity of all cuts.
  |f|= f(S,T) i.e. the value of a max flow is equal to the capacity of a min cut.

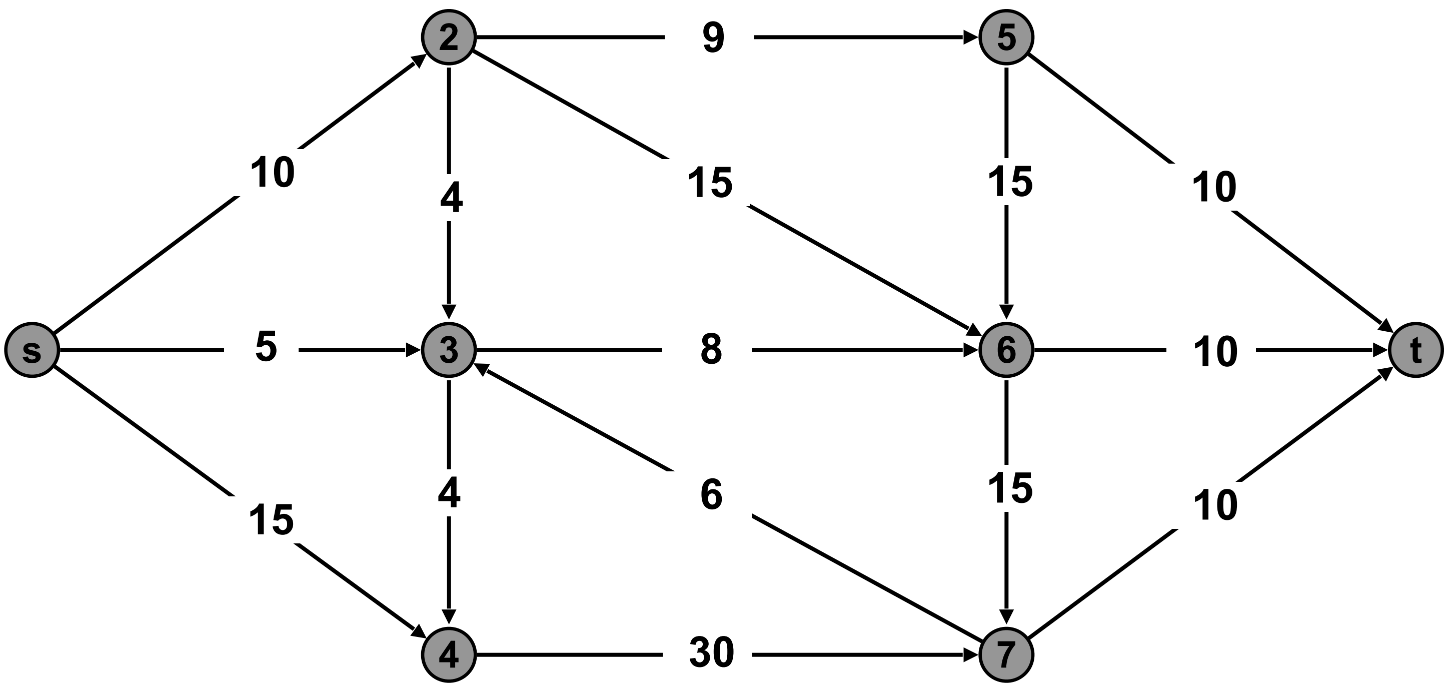


# Max Flow Network

_G = (V, E, s, t, u) _(V, E) = directed graph, no parallel arcs.

_Two distinguished nodes:  s = source, t = sink.

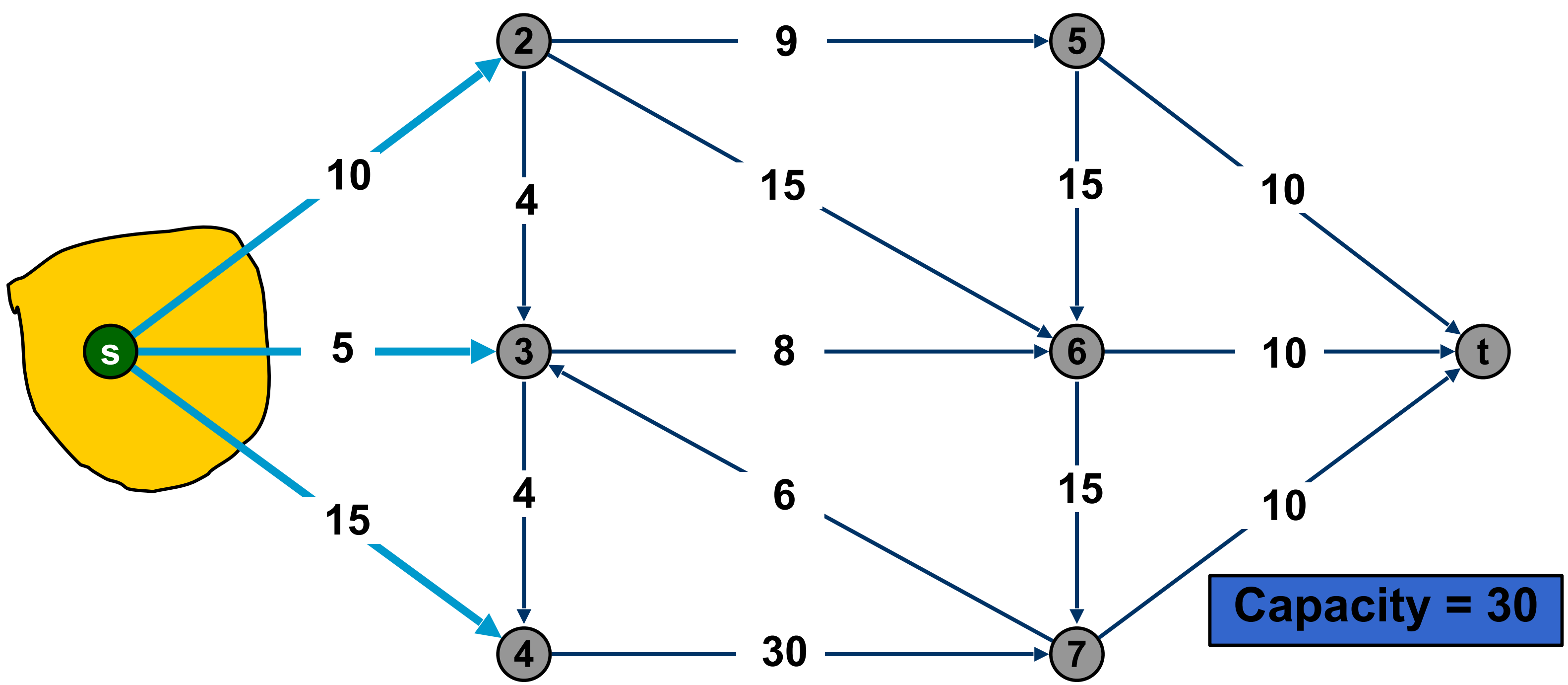
_u(e) = capacity of arc e.




**MAX FLOW:  find s-t flow that maximizes net flow out of the source.**

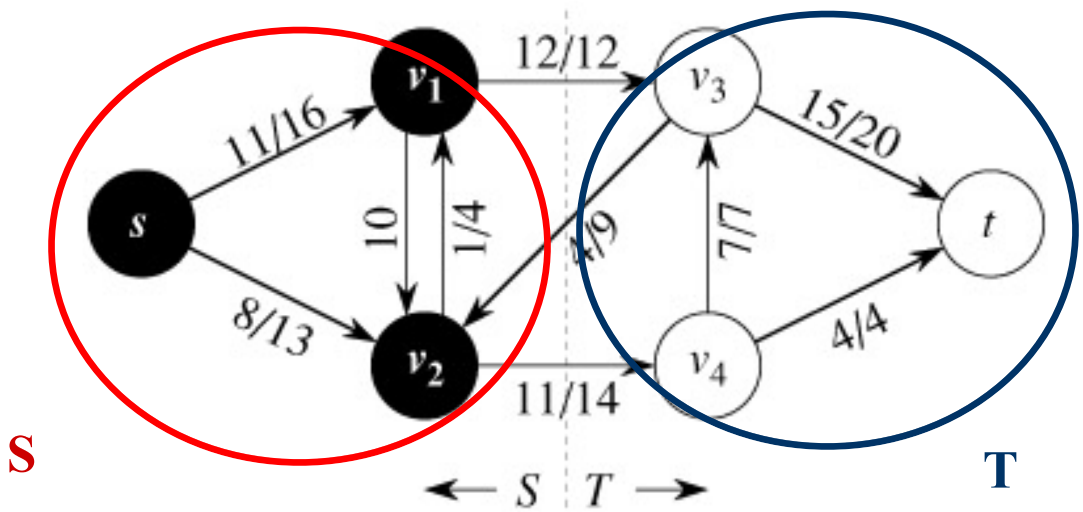
# Cuts of Flow Networks

- A cut in a network is a partition of V into S and T=V-S so that s is in S and t is in T.



# The Net Flow Through a Cut(S,T)

$$f(S,T) = \sum_{u \in S, v \in T} f(u,v) - \sum_{v \in T, u \in S} f(v,u)$$

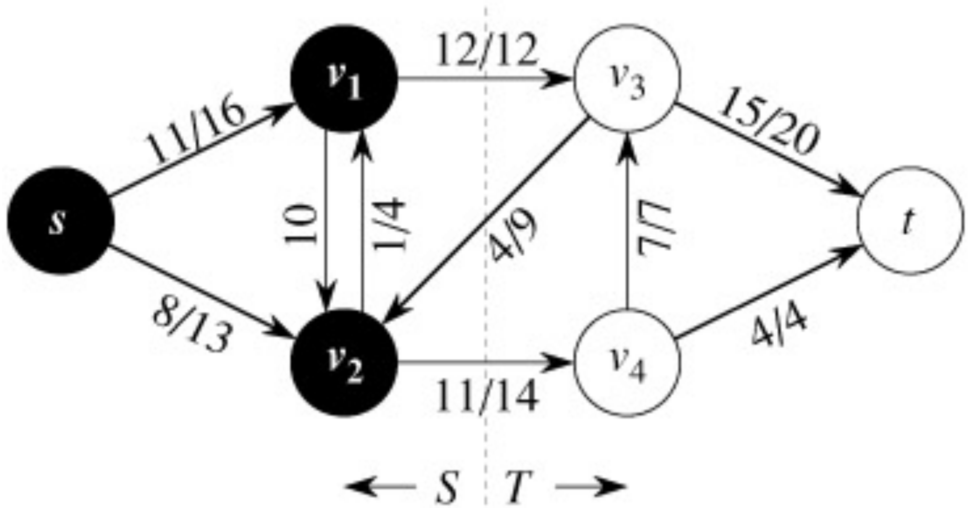



$f(S,T) = 12 – 4 + 11 = 19$

**The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.**

# The Capacity of Cut(S,T)

$$\boxed{c(S,T) = \sum_{u \in S, v \in T} c(u,v)}$$




c(S,T)= 12+ 14 = 26

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Always nonnegative (why?)

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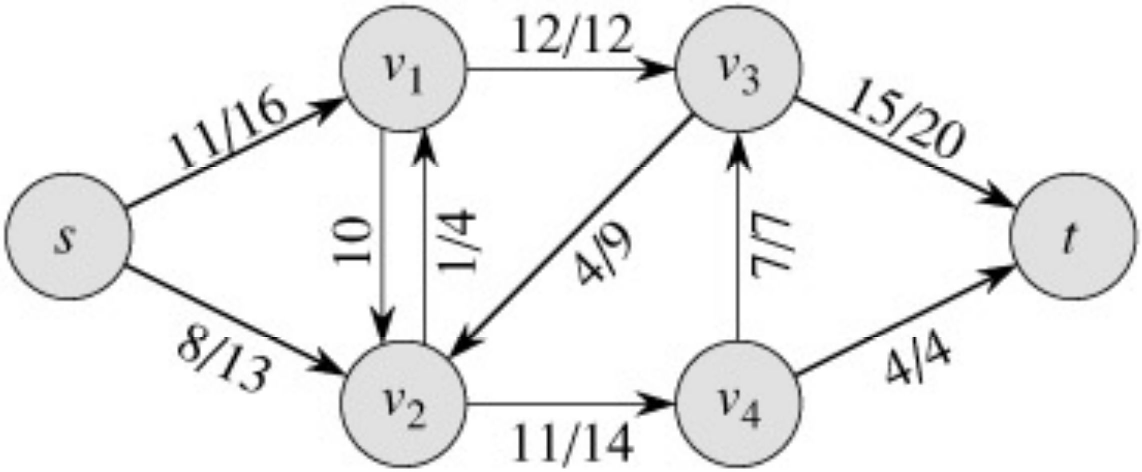

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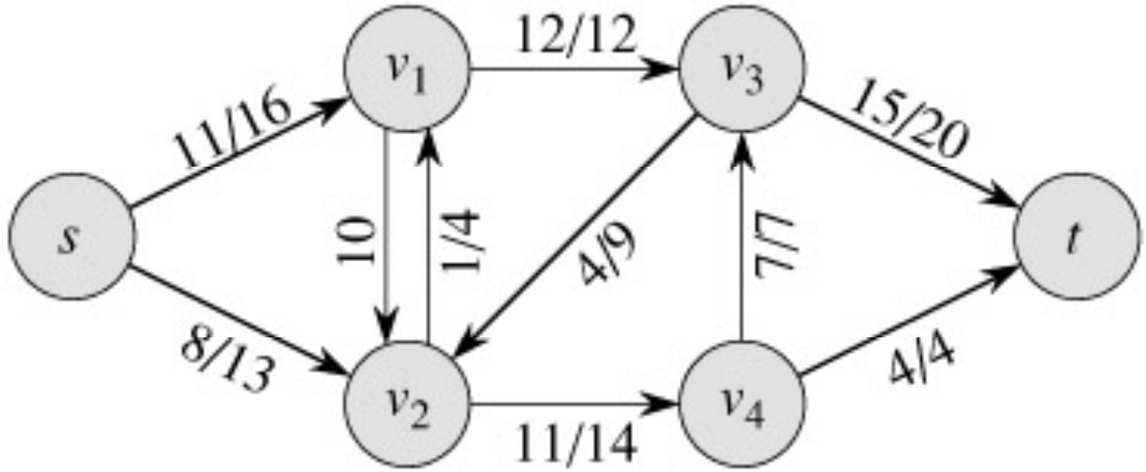
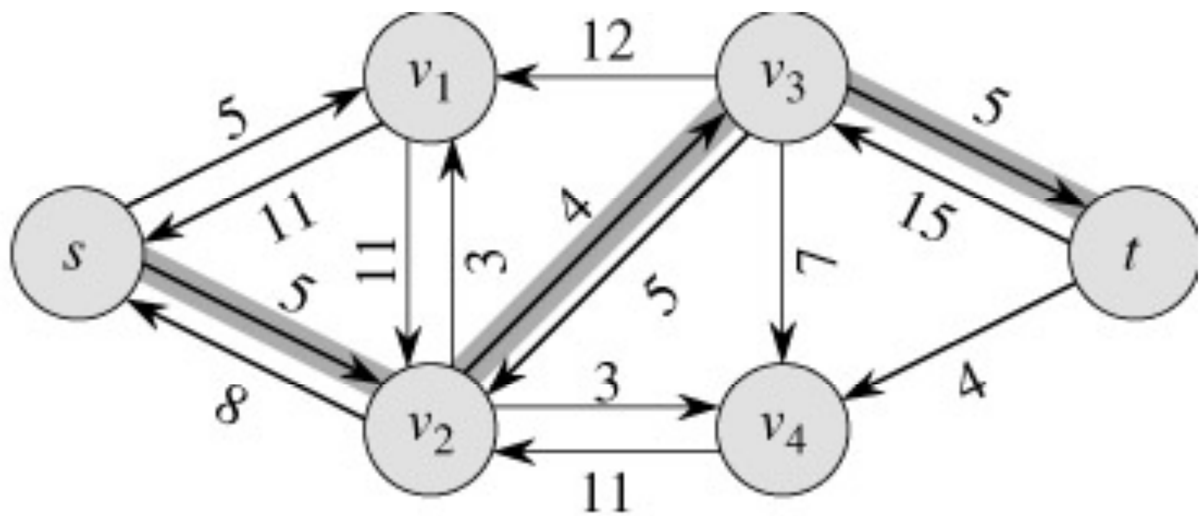



Residual Network:



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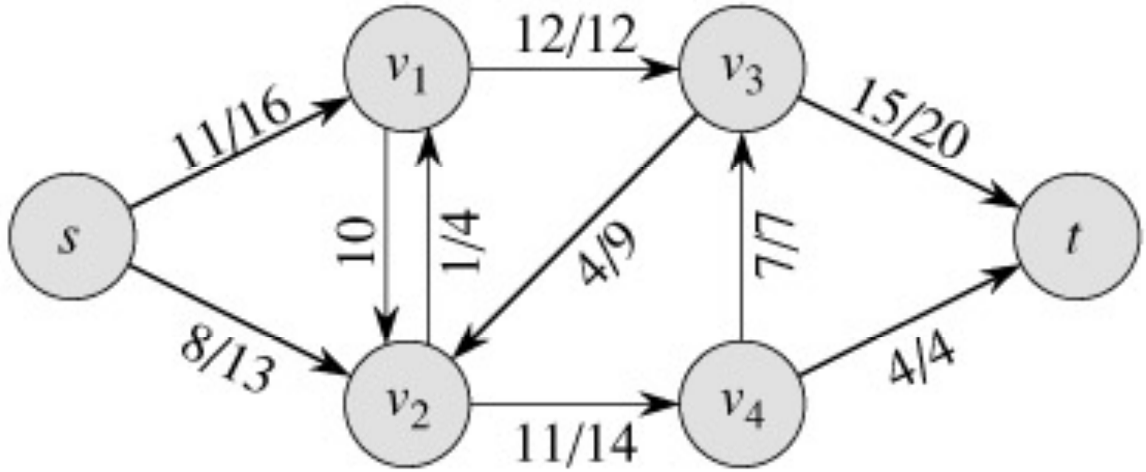


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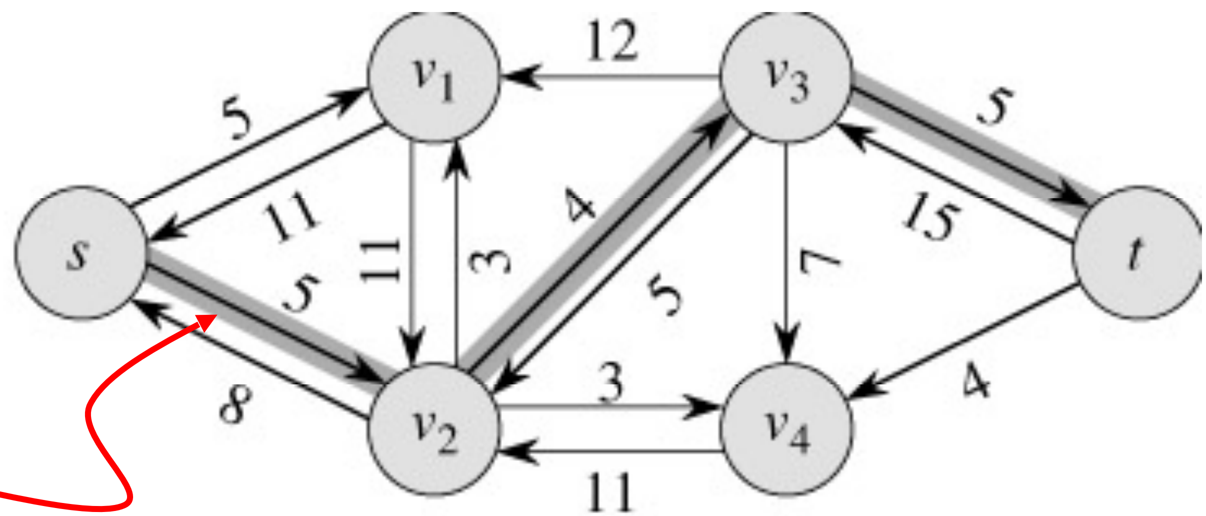
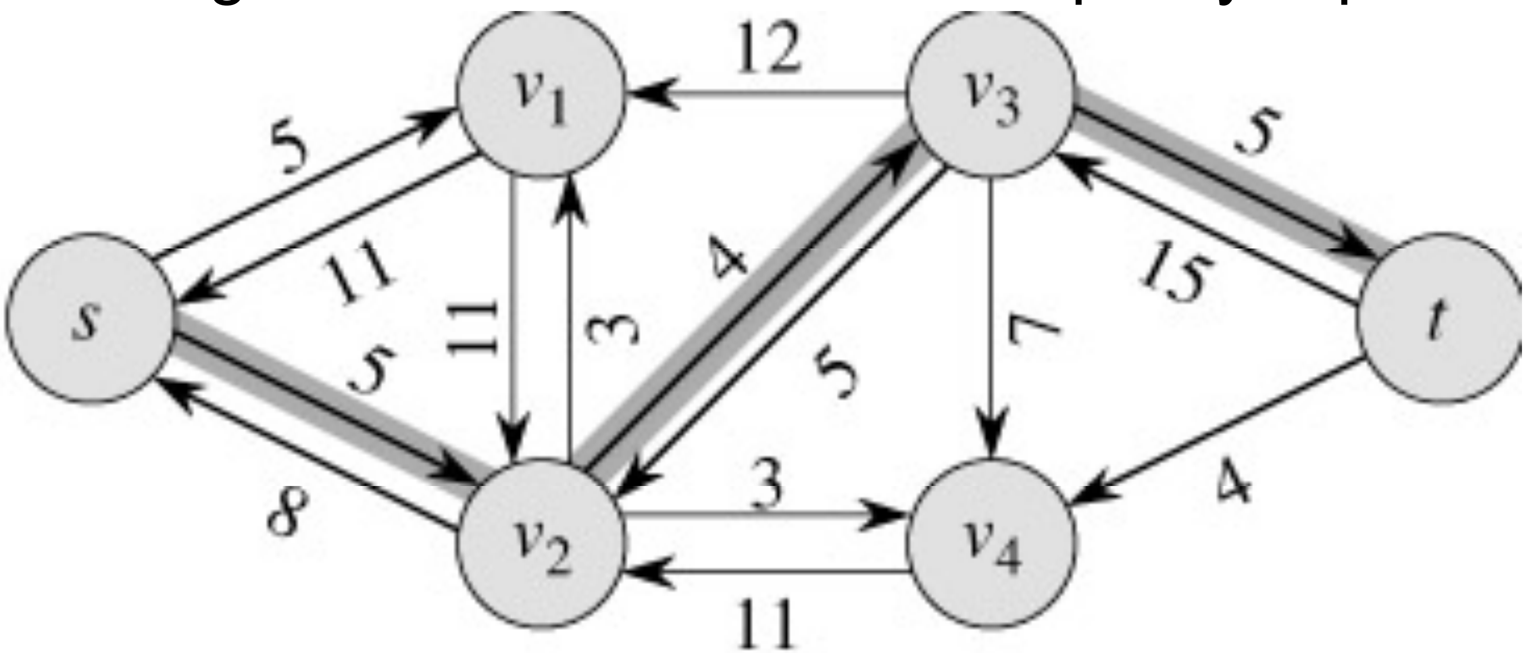



Augmenting path

# The residual network

- The edges of the residual network are the edges on which the residual capacity is positive.



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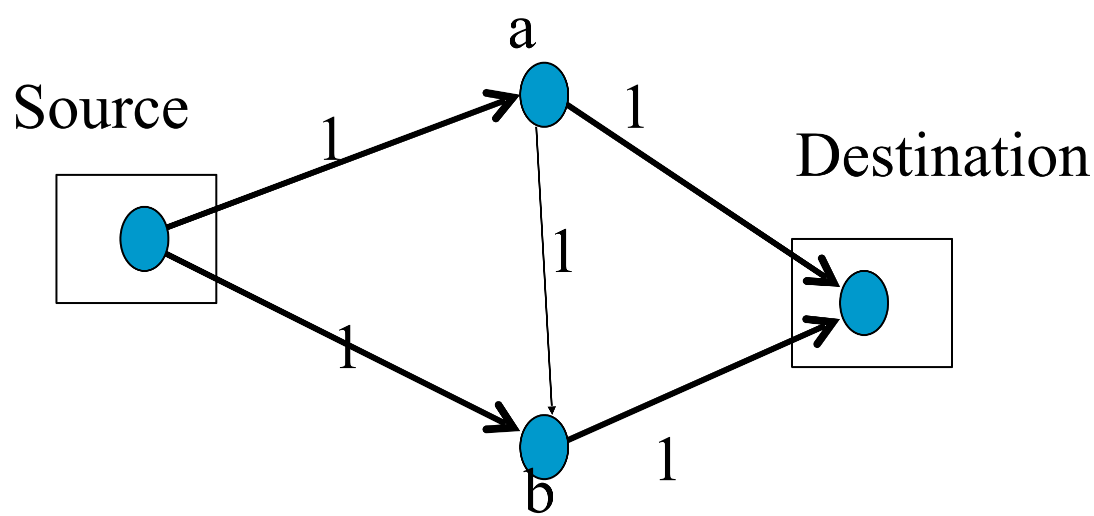
- If we have taken a bad path then residual networks allow one to detect the condition and reverse the flow.
- A bad path is one which overlaps with too many other paths.

# Example



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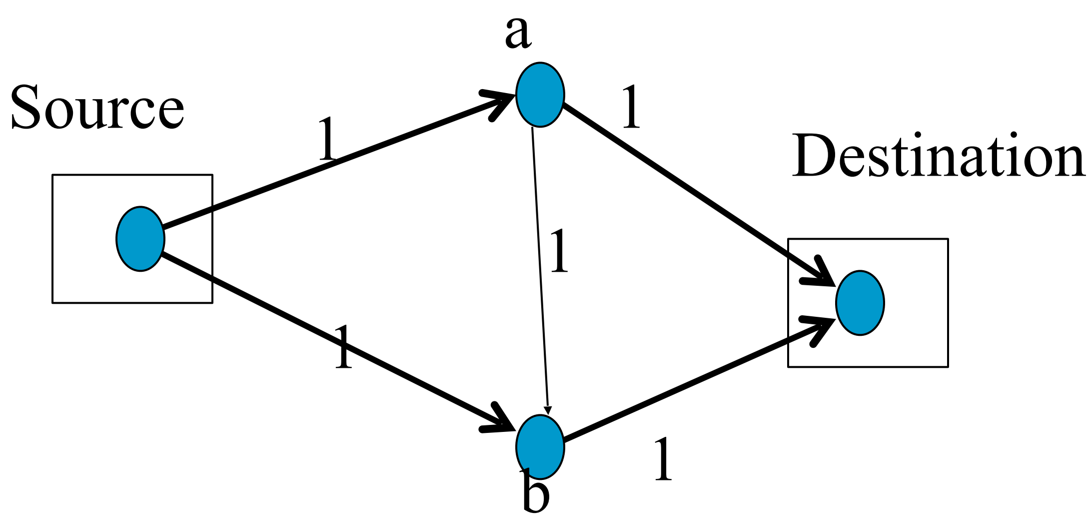


Paths source, a, destinations and source, b destination gives a flow of 2 units.

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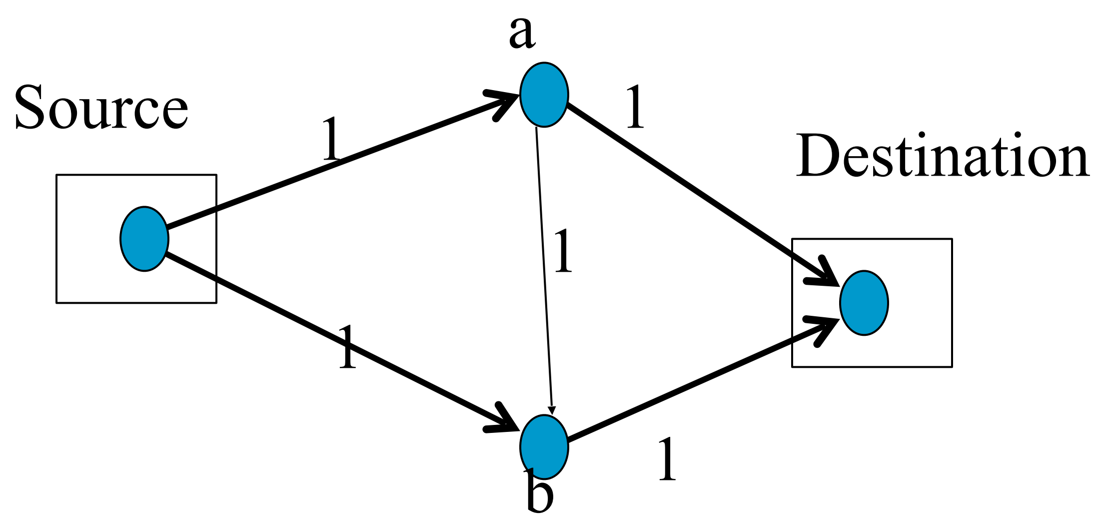


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Path source, a, b, destination overlaps with both the optimal paths.

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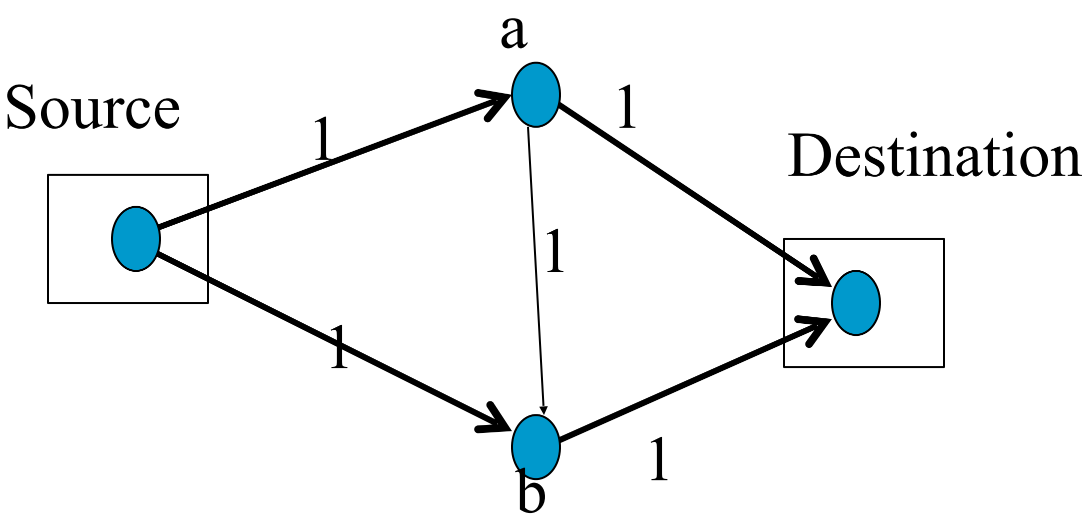


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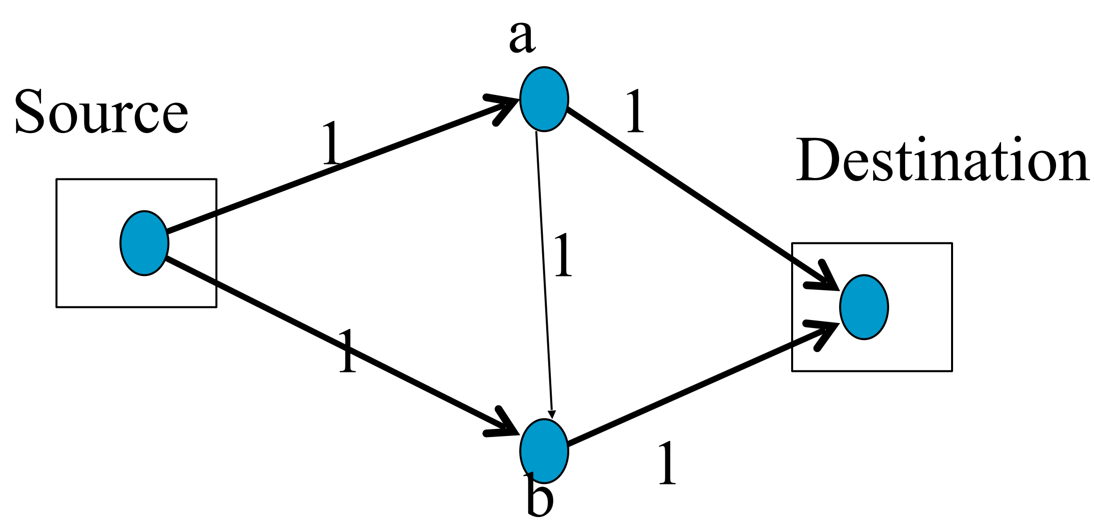


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Verify how we recover in spite of the initial bad choice, if we use the residual network to augment flows.

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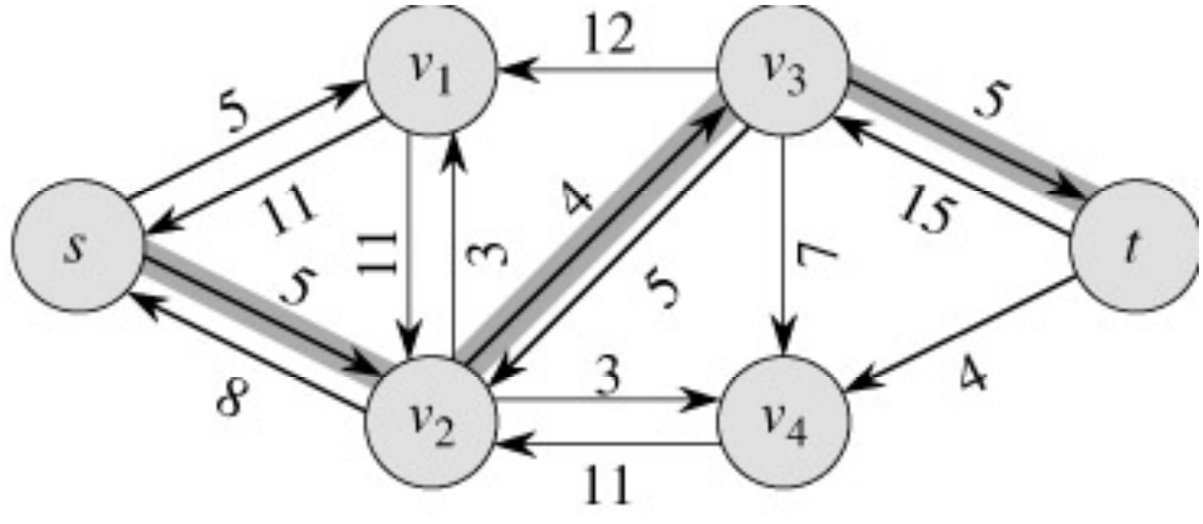
$$c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}$$

# Augmenting Paths - example



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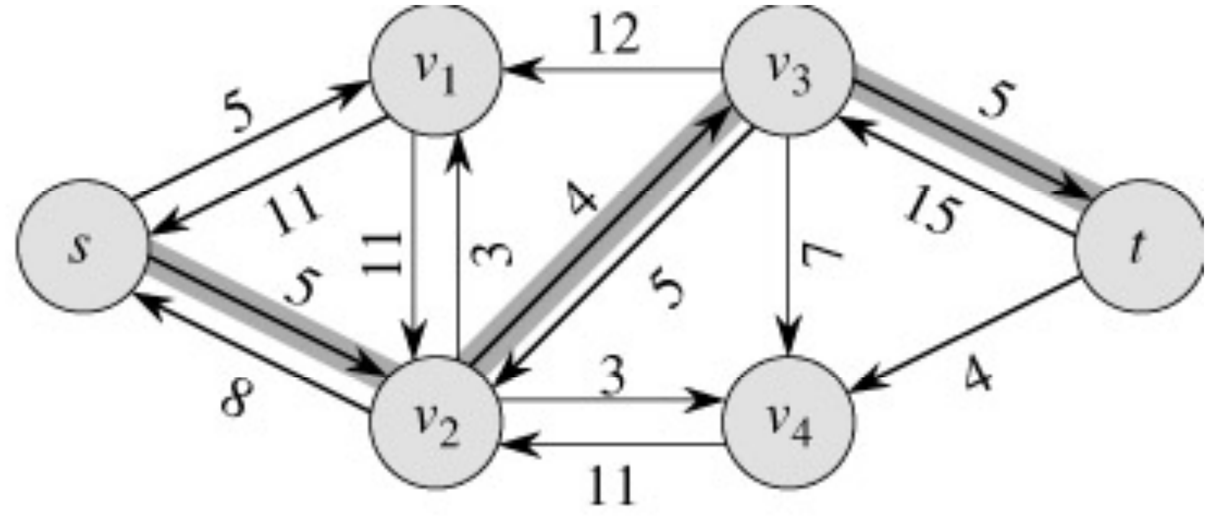


- The residual capacity of p = 4.

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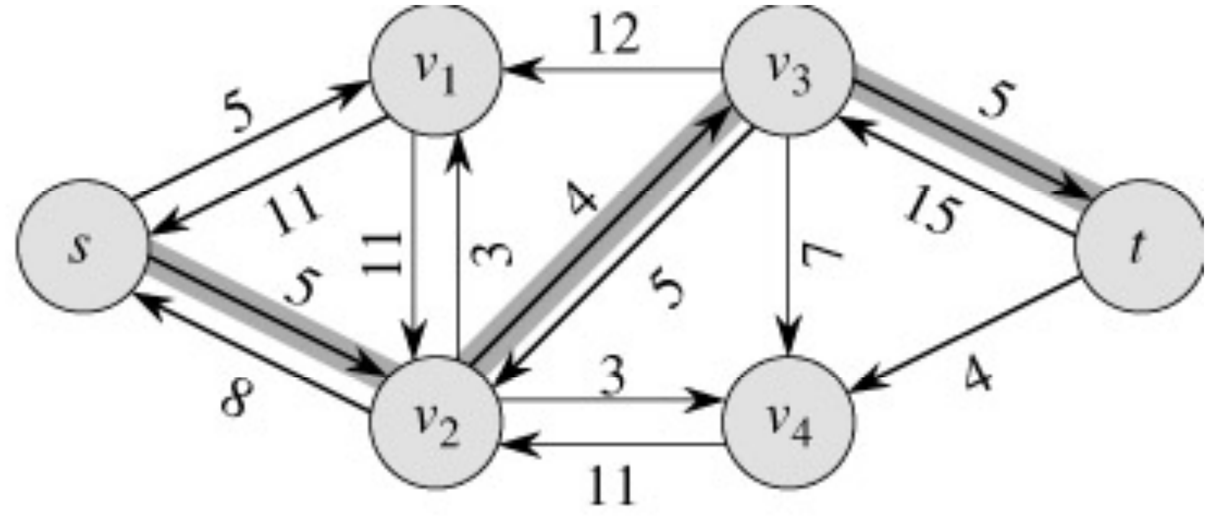



- The residual capacity of p = 4.
- Can improve the flow along p by 4.

# Maxflow-Mincut Theorem

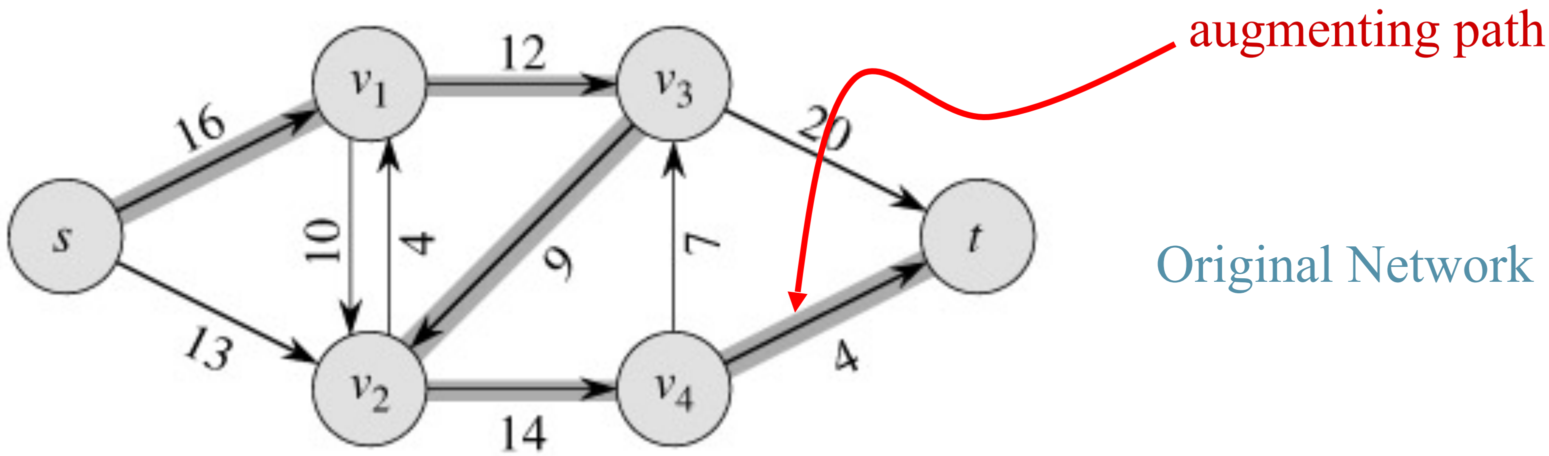
- *Max-flow min-cut theorem*:
  - If $f$ is the flow in G, the following conditions are equivalent:
    - 1. $f$ is a maximum flow in $G$
    - 2. The residual network $G_f$ contains no augmenting paths
    - 3. $|f| = c(S,T)$ for some cut $(S,T)$ of $G$

# Ford-Fulkerson Method

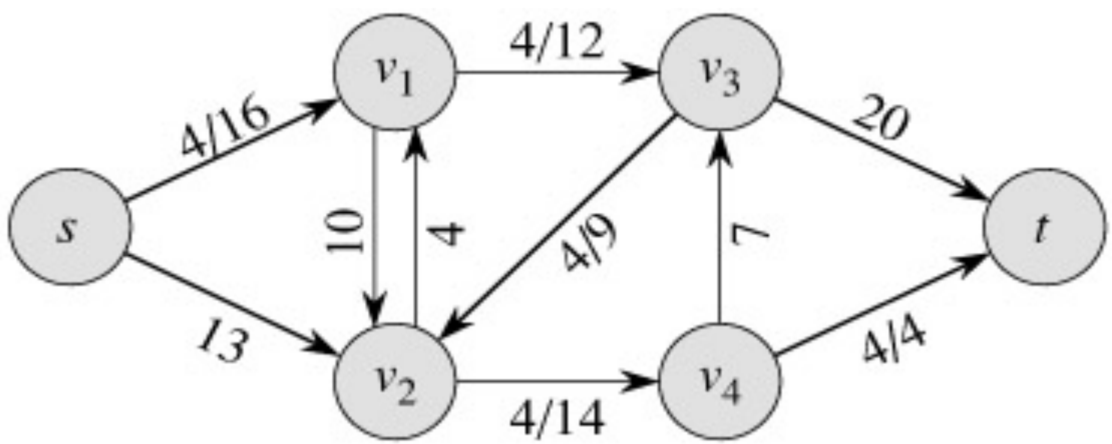
FORD-FULKERSON-METHOD$(G, s, t)$

initialize flow $f$ to 0<br>
**while** there exists an augmenting path $p$<br>
    **do** augment flow $f$ along $p$<br>
**return** $f$

# Example

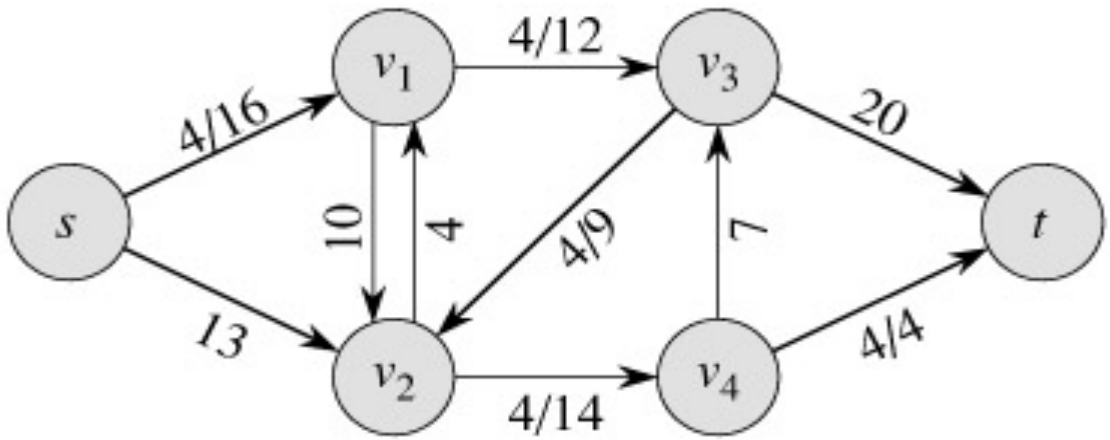



Original Network



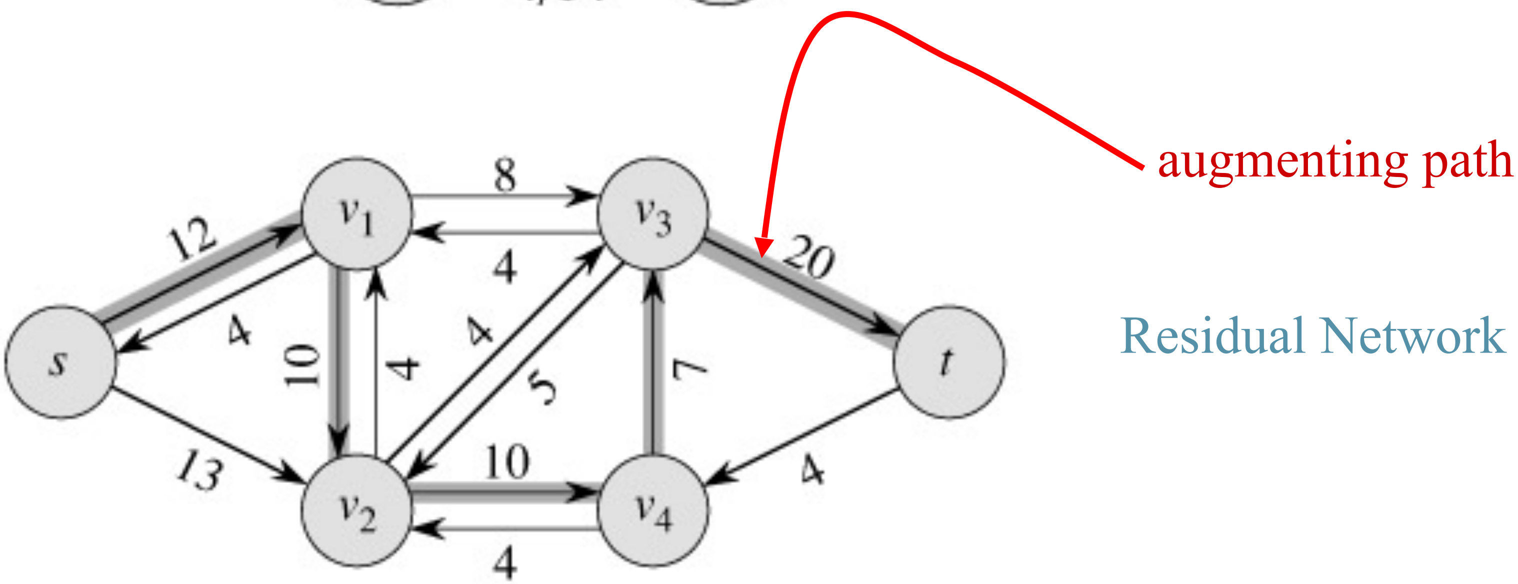

Resulting Flow =4

# Example




Resulting

Flow = 4




Residual Network

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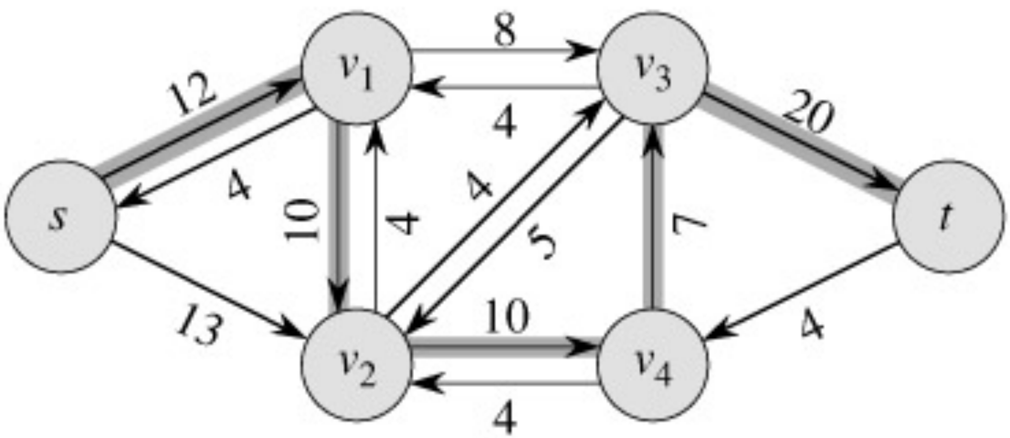


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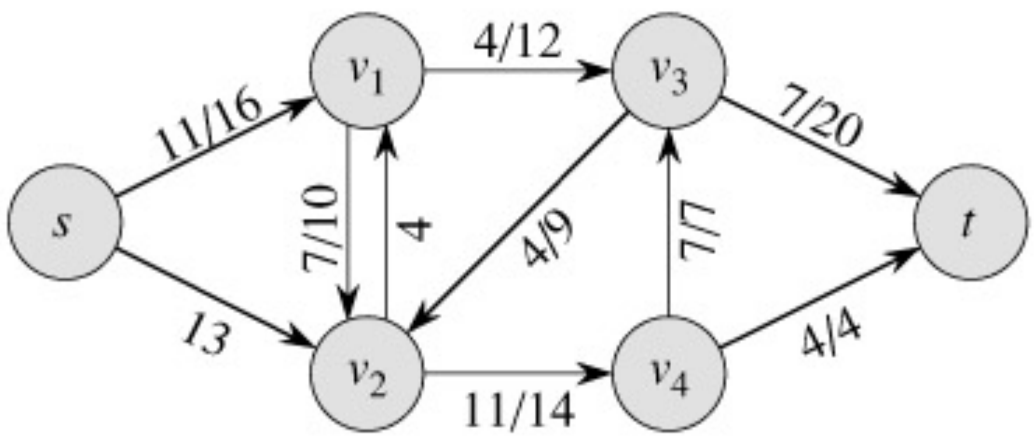
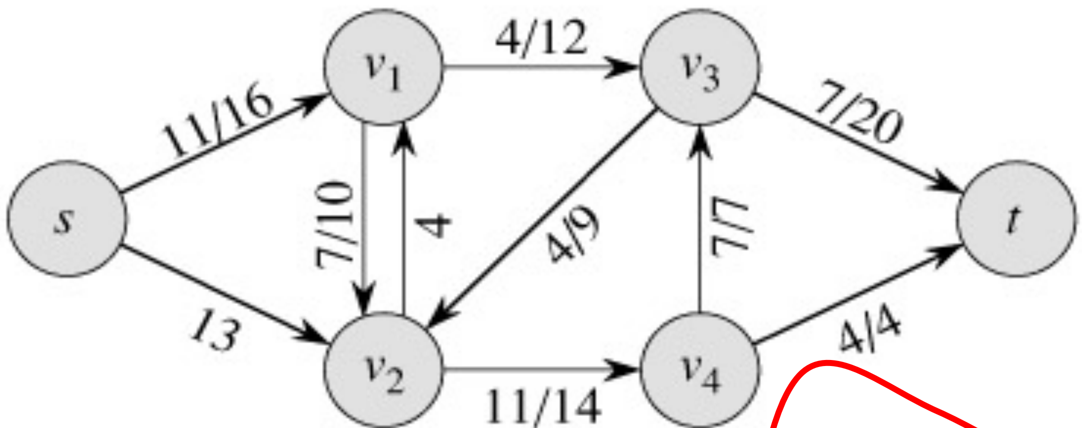



Resulting Flow =11

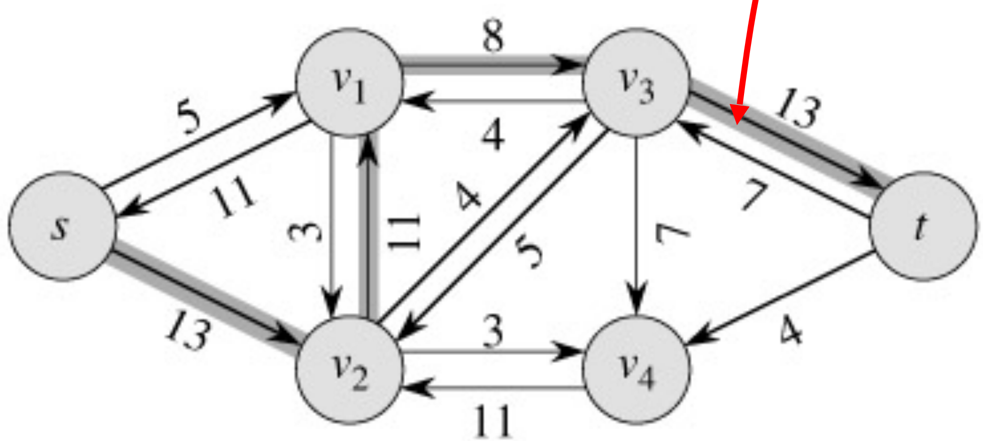
# Example




Resulting

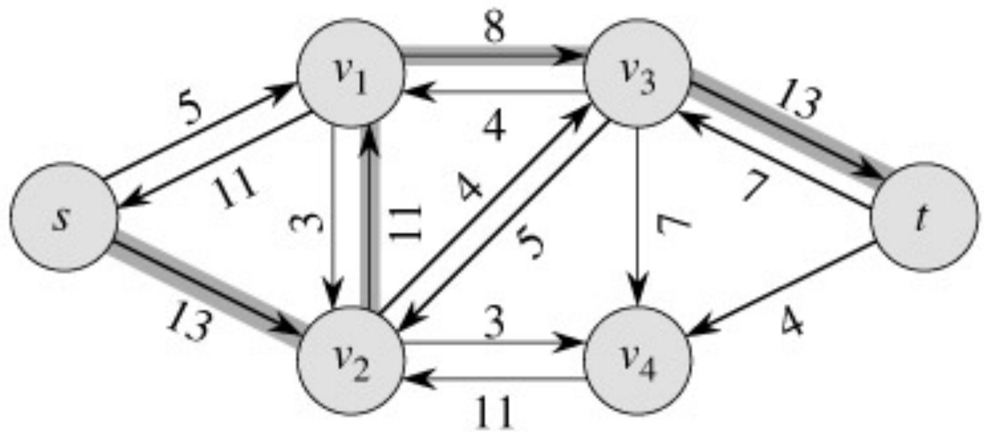
Flow =11

augmenting path

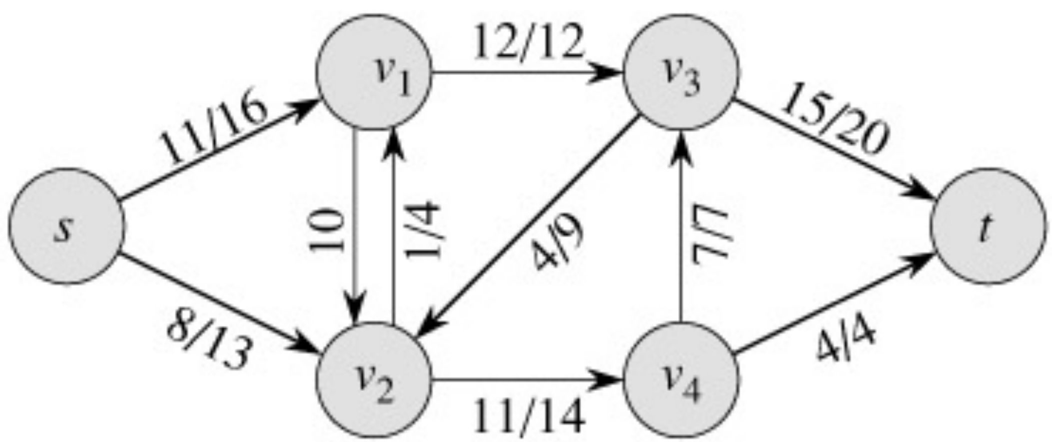



Residual Network

# Example



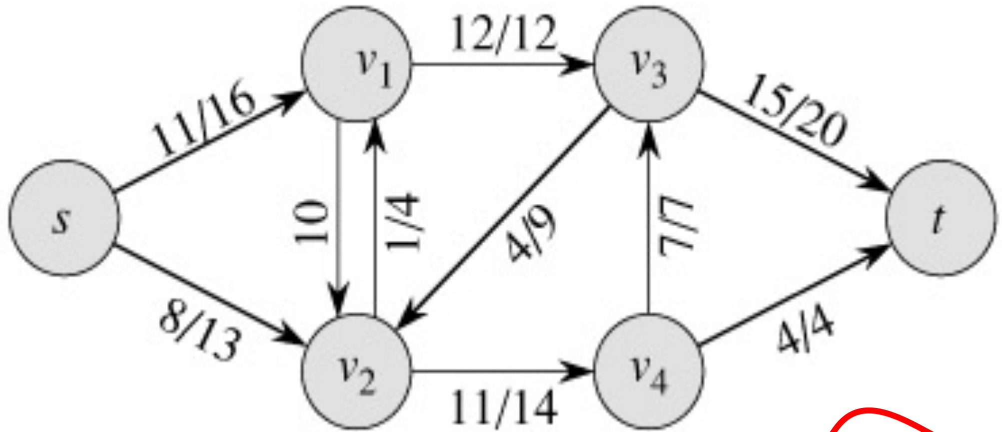

Residual Network




Resulting

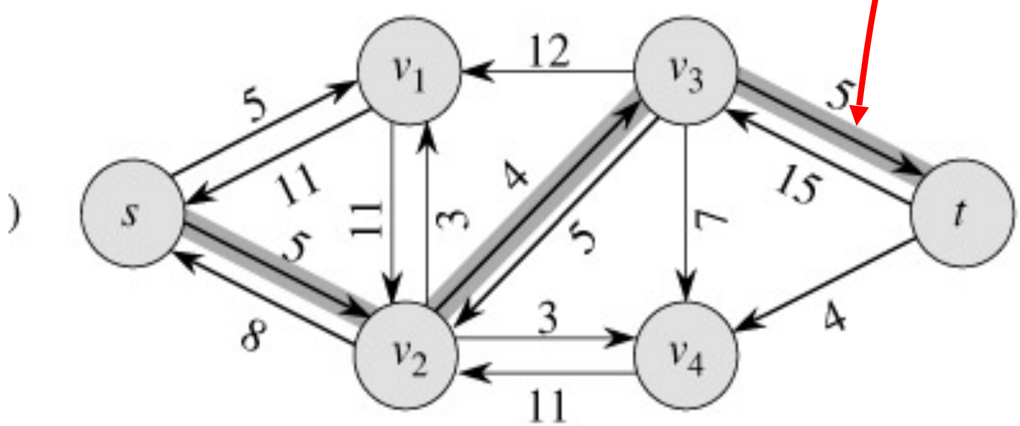
Flow =19

# Example




Resulting

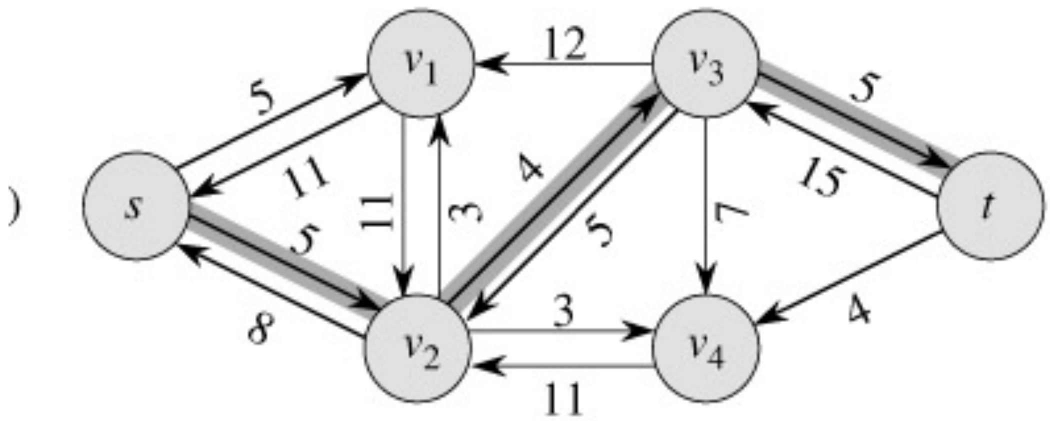
Flow = 19



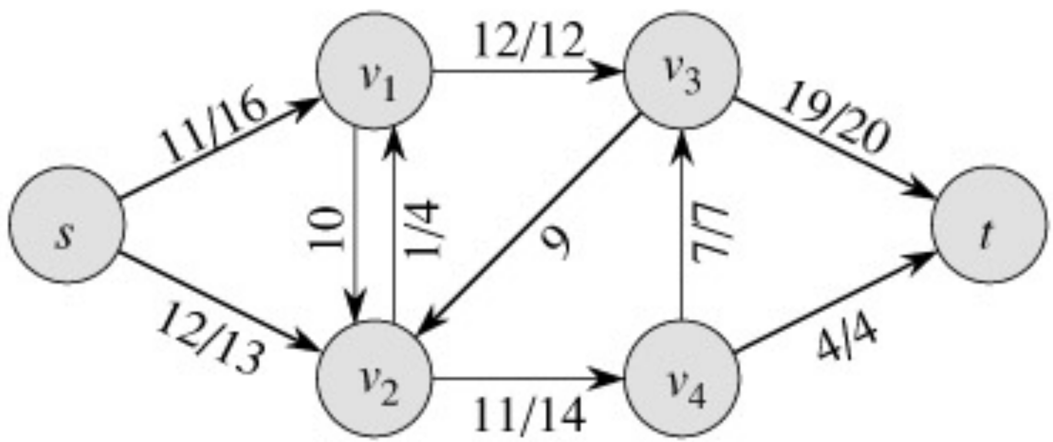

augmenting path

Residual Network

# Example



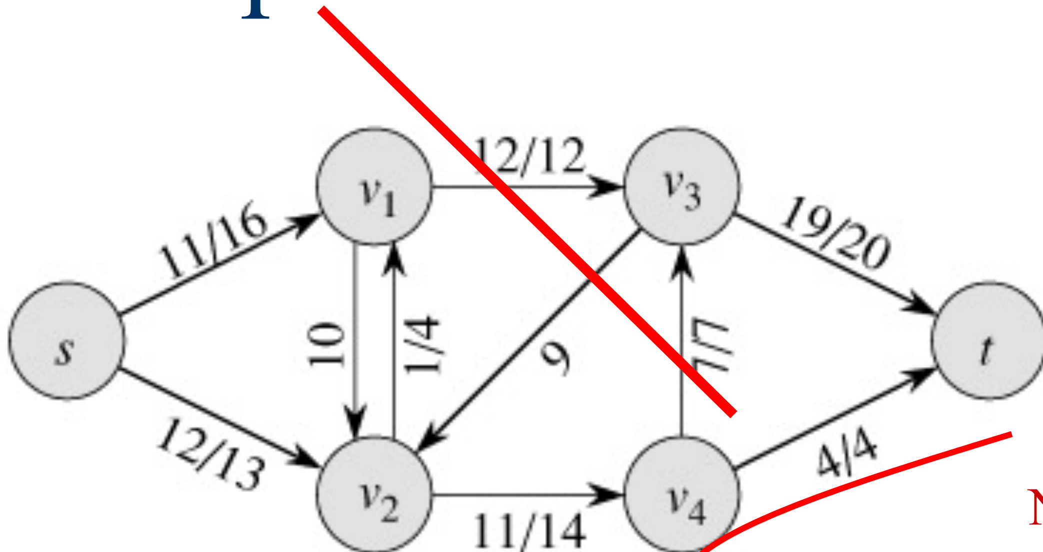

Residual Network




Resulting

Flow = 23

# Example



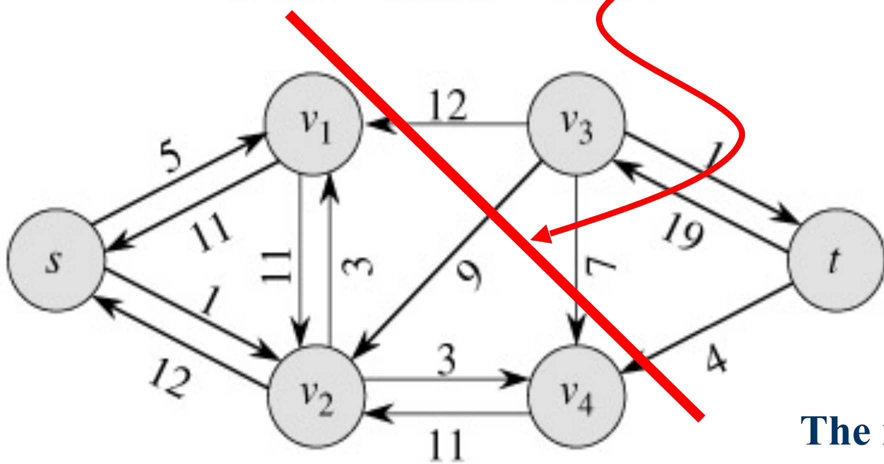

Resulting

Flow = 23

No augmenting path:

Maxflow=23

(e)




Residual Network

**The residual network Gf contains no augmented paths. So f is a maximum flow in G.**

# Ford-Fulkerson method, with details

```
Ford-Fulkerson(G,s,t)
for each edge (u,v)∈G.E do
   f(u,v) = f(v,u) = 0
while ∃ path p from s to t in residual network G_f do
   c_f = min{c_f(u,v): (u,v)∈p}
   for each edge (u,v) in p do
      f(u,v) = f(u,v) + c_f
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return f
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7          f(v,u) = -f(u,v)
8  return f
```

The algorithms based on this method differ in how they choose *p* in step 3.

# Time Analysis I

- A complete analysis establishing which specific method is best is a complex task, however, because their running times depend on
  - The number of augmenting paths needed to find a maxflow
  - The time needed to find each augmenting path

# Analysis

FORD-FULKERSON$(G, s, t)$

```
for each edge (u, v) ∈ E[G]
        do f[u, v] ← 0
           f[v, u] ← 0
while there exists a path p from s to t in the residual network G_f
        do c_f(p) ← min {c_f(u, v) : (u, v) is in p}
           for each edge (u, v) in p
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                   f[v, u] ← −f[u, v]
```

O(E)

# Analysis

```
FORD-FULKERSON(G, s, t)
for each edge (u, v) ∈ E[G]
     do f[u, v] ← 0                                   O(E)
        f[v, u] ← 0
while there exists a path p from s to t in the residual network G_f
     do c_f(p) ← min {c_f(u, v) : (u, v) is in p}
        for each edge (u, v) in p
            do f[u, v] ← f[u, v] + c_f(p)                    O(E)
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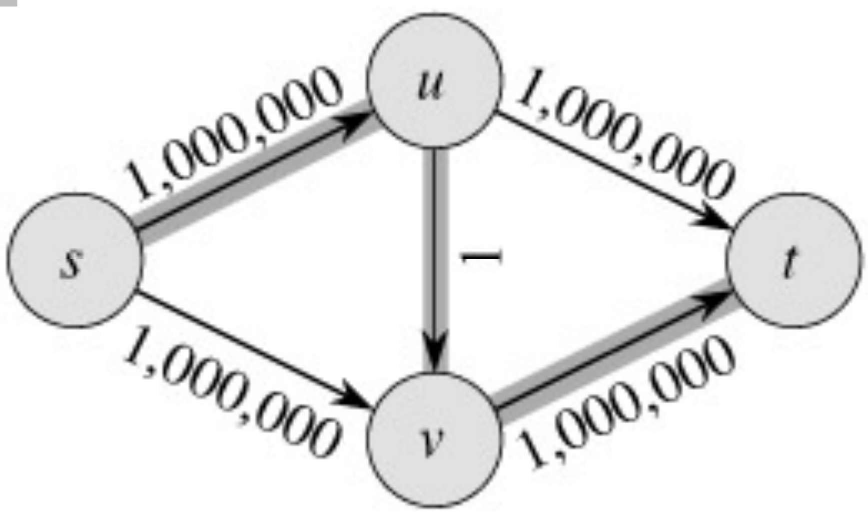
- Note that this running time is not polynomial in input size. It depends on $|f^*|$, which is not a function of $|V|$ or $|E|$.
- If capacities are rational, can scale them to integers.
- If irrational, FORD-FULKERSON might never terminate!

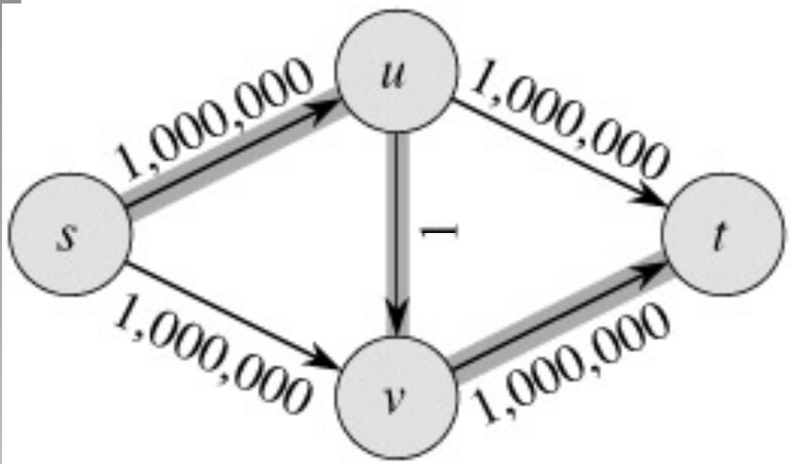
# The basic Ford-Fulkerson Algorithm

- With time O ( E |f*|), the algorithm is not polynomial.
- This problem is real: Ford-Fulkerson may perform very badly if we are unlucky:

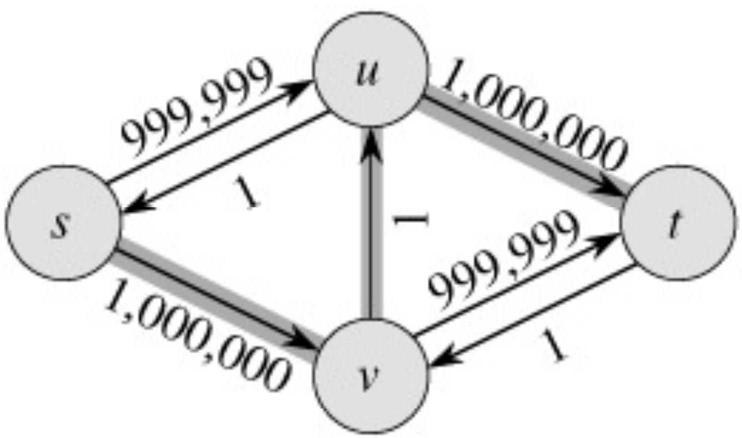



|f*|=2,000,000

# Run Ford-Fulkerson on this example




Augmenting Path




Residual Network

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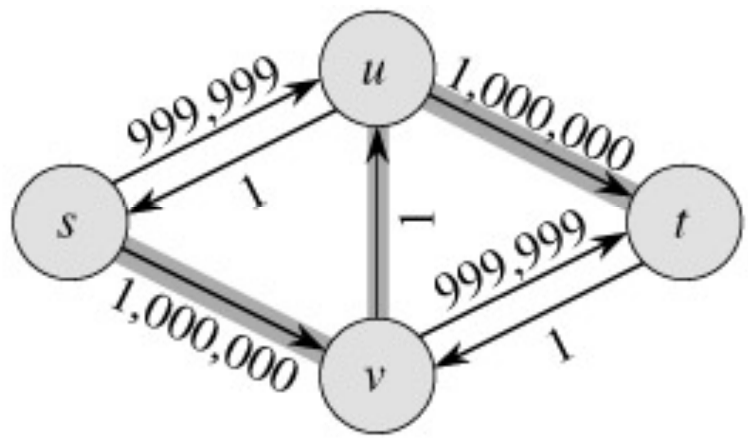
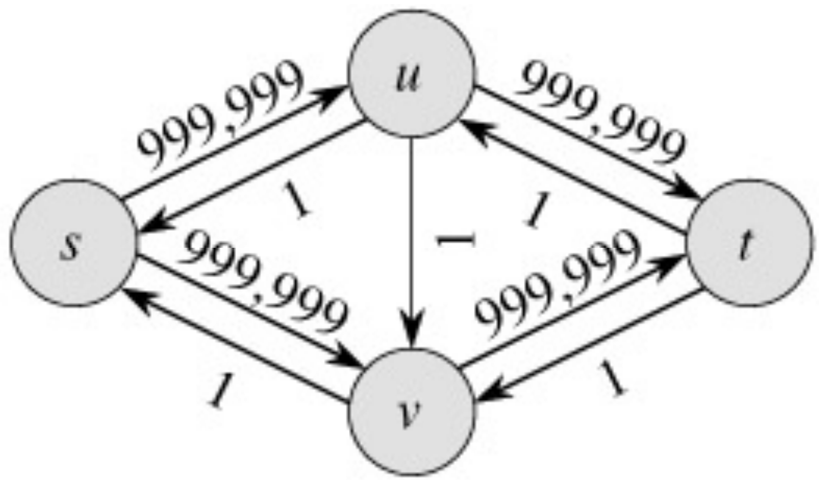
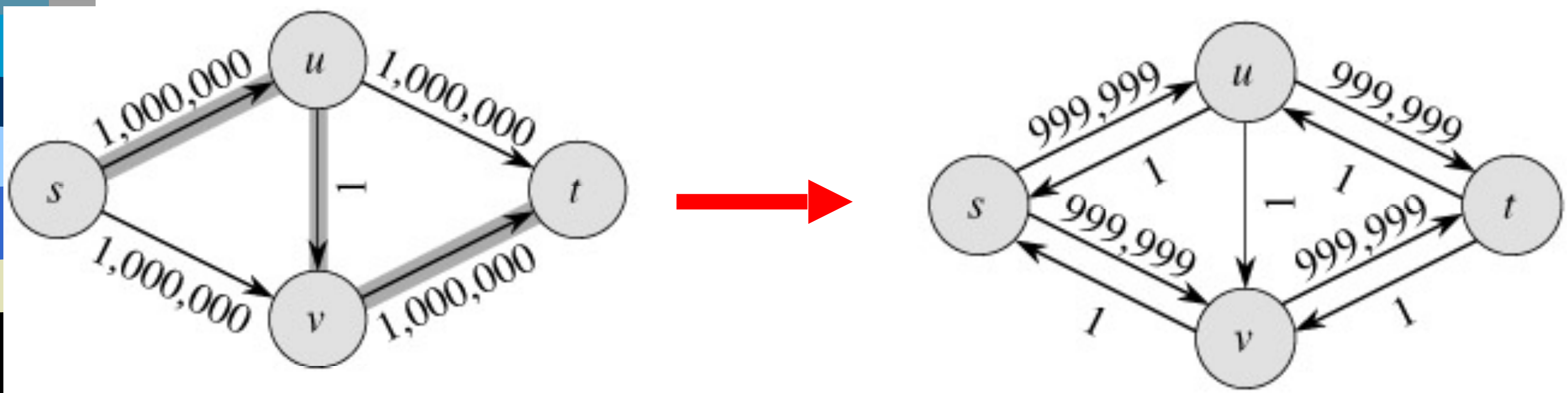



Augmenting Path




Residual Network

# Run Ford-Fulkerson on this example




- Repeat 999,999 more times…

# The Edmonds-Karp Algorithm

FORD-FULKERSON$(G, s, t)$

**for** each edge $(u, v) \in E[G]$
  **do** $f[u, v] \leftarrow 0$
    $f[v, u] \leftarrow 0$
**while** there exists a path $p$ from $s$ to $t$ in the residual network $G_f$
  **do** $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
    **for** each edge $(u, v)$ in $p$
      **do** $f[u, v] \leftarrow f[u, v] + c_f(p)$
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# The Edmonds-Karp Algorithm

- A small fix to the Ford-Fulkerson algorithm makes it work in polynomial time.

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# The Edmonds-Karp Algorithm

- A small fix to the Ford-Fulkerson algorithm makes it work in polynomial time.
- Specify how to compute the path in line 4.

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2     **do** $f[u, v] \leftarrow 0$
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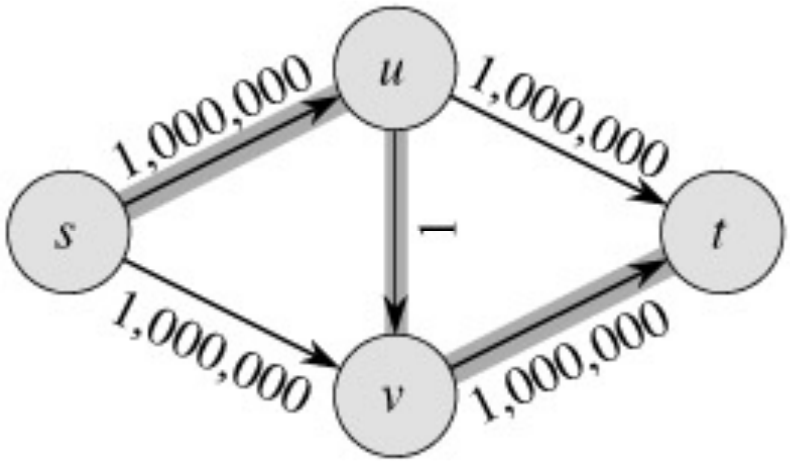
- Compute the path in line 4 using breadth-first search on residual network.
- The augmenting path p is the shortest path from s to t in the residual network (treating all edge weights as 1).
- Runs in time $O(V E^2)$.

# The Edmonds-Karp Algorithm - example




- Edmonds-Karp's algorithm runs only 2 iterations on this graph.

# Time Complexity

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    do c_f(p) ← min {c_f(u, v) : (u, v) is in p}
       for each edge (u, v) in p
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- Let, total number of flow augmentations performed by Edmonds-Karp algorithm is O(VE)

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```

- Let, total number of flow augmentations performed by Edmonds-Karp algorithm is O(VE)
- BFS to find the augmented path – O(E)
- So, Total running time is $O(VE^2)$

# Conditions

If f is a flow in a flow network G=(V,E), with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network $G_f$ contains no augmented paths.
3. $|f| = c(S,T)$ for some cut (S,T) (a min-cut).

It is a flow since there is no augmented paths It is maximum since the sink is not reachable from the source