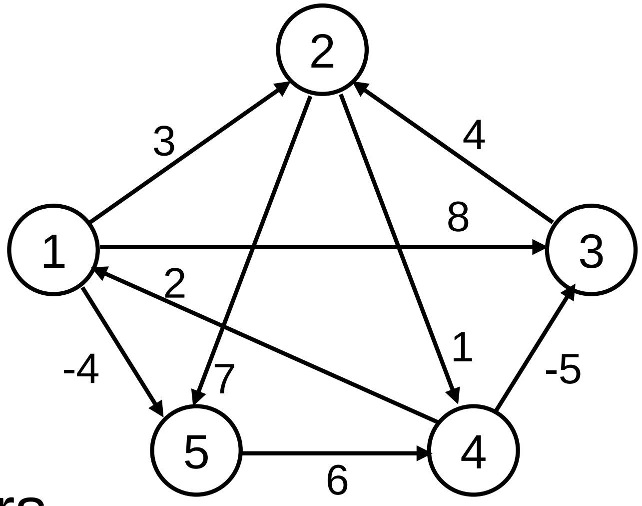
# Combinatorial Optimization
# CSE 301

## All Pairs of Shortest Path

# All-Pairs Shortest Paths

- **Given:**
  - Directed graph G = (V, E)
  - Weight function w : E → **R**
- **Compute:**
  - The shortest paths between all pairs of vertices in a graph
  - Representation of the result: an n × n matrix of shortest-path distances δ(u, v)



# Dijkstra (G, w, s)

```
INITIALIZE-SINGLE-SOURCE(V, s)  ⟵ Θ(V)
S ← ∅
Q ← V[G]  ⟵ O(V) build min-heap
while Q ≠ ∅  ⟵ Executed O(V) times
    do u ← EXTRACT-MIN(Q)  ⟵ O(lgV)
        S ← S ∪ {u}
        for each vertex v ∈ Adj[u]
            do RELAX(u, v, w)  ⟵ O(E) times; O(lgV)
```

Running time: $O(V\lg V + E\lg V) = O(E\lg V)$

# BELLMAN-FORD(*V*, *E*, *w*, *s*)

1. INITIALIZE-SINGLE-SOURCE(V, s) ← $\Theta(V)$
2. **for** i ← 1 to |V| - 1 ← O(V)
3. **do for** each edge (u, v) ∈ E ← O(E)
4. **do** RELAX(u, v, w)

(lines 2–4: O(VE))

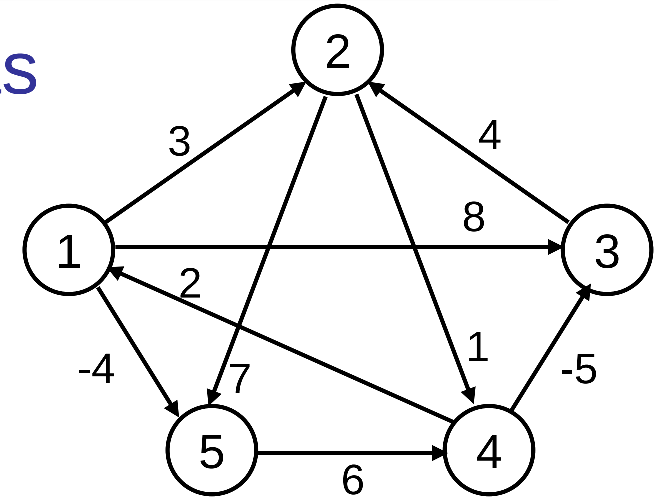
5. **for** each edge (u, v) ∈ E ← O(E)
6. **do if** d[v] > d[u] + w(u, v)
7. **then return** FALSE
8. **return** TRUE

Running time: O(VE)

# All-Pairs Shortest Paths - Solutions

- Run **BELLMAN-FORD** once from each vertex:
  - $O(V^2E)$, which is $O(V^4)$ if the graph is dense ($E = \Theta(V^2)$)
- If no negative-weight edges, could run **Dijkstra's** algorithm once from each vertex:
  - $O(VE\lg V)$ with binary heap, $O(V^3\lg V)$ if the graph is dense
- We can solve the problem in $O(V^3)$, with no elaborate data structures

# All-Pairs Shortest Paths




- Assume the graph (G) is given as adjacency matrix of weights
  - $W = (w_{ij})$, $n \times n$ matrix, $|V| = n$
  - Vertices numbered 1 to $n$

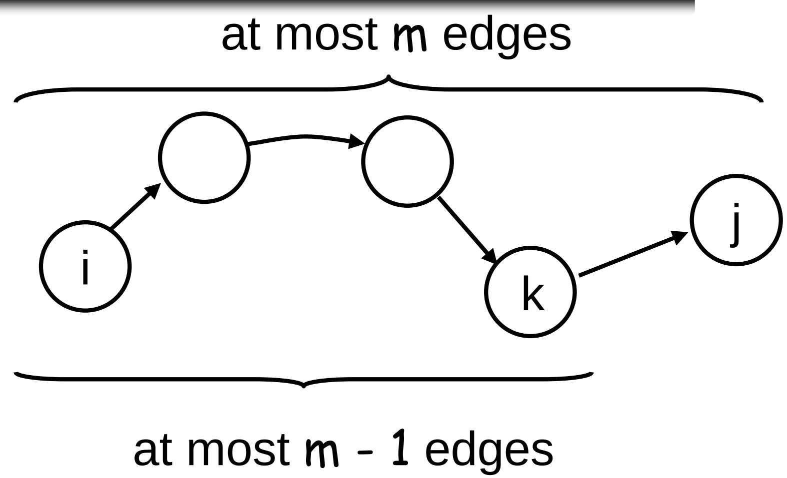
$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{weight of } (i, j) & \text{if } i \neq j,\ (i, j) \in E \\ \infty & \text{if } i \neq j,\ (i, j) \notin E \end{cases}$$

- Output the result in an $n \times n$ matrix

  $D = (d_{ij})$, where $d_{ij} = \delta(i, j)$
- Solve the problem using dynamic programming

# Optimal Substructure of a Shortest Path

- All subpaths of a shortest path are shortest paths
- Let p: a shortest path p from vertex i to j that contains at most m edges
- If i = j
  - w(p) = 0 and p has no edges

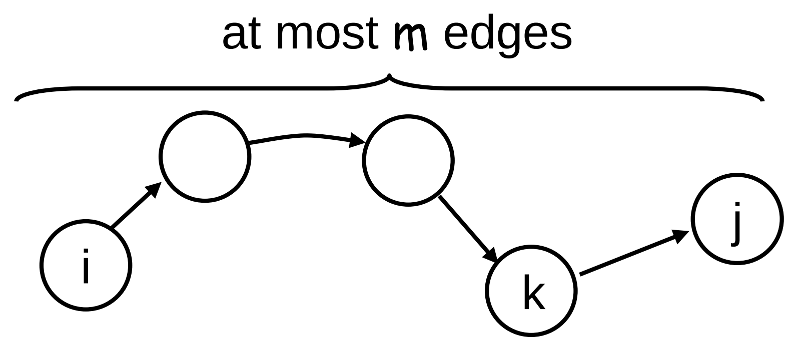



- If $i \neq j$: $p = i \overset{p'}{\rightsquigarrow} k \rightarrow j$
  - p′ has at most m-1 edges
  - p′ is a shortest path

$$\delta(i, j) = \delta(i, k) + w_{kj}$$

# Recursive Solution

- $l_{ij}^{(m)}$ = weight of shortest path $i \rightsquigarrow j$ that contains at most $m$ edges




- $m = 0$: $l_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases}$

- $m \geq 1$: $l_{ij}^{(m)} = \min \{ l_{ij}^{(m-1)} , \min_{1 \leq k \leq n} \{ l_{ik}^{(m-1)} + w_{kj} \} \}$

  $= \min_{1 \leq k \leq n} \{ l_{ik}^{(m-1)} + w_{kj} \}$

  – Shortest path from i to j with at most m – 1 edges

  – Shortest path from i to j containing at most m edges, considering all possible predecessors (k) of j

# Computing the Shortest Paths

- m = 1: $l_{ij}^{(1)} = w_{ij}$ $L^{(1)} = W$
  - The path between i and j is restricted to 1 edge
- Given $W = (w_{ij})$, compute: $L^{(1)}, L^{(2)}, \ldots, L^{(n-1)}$, where

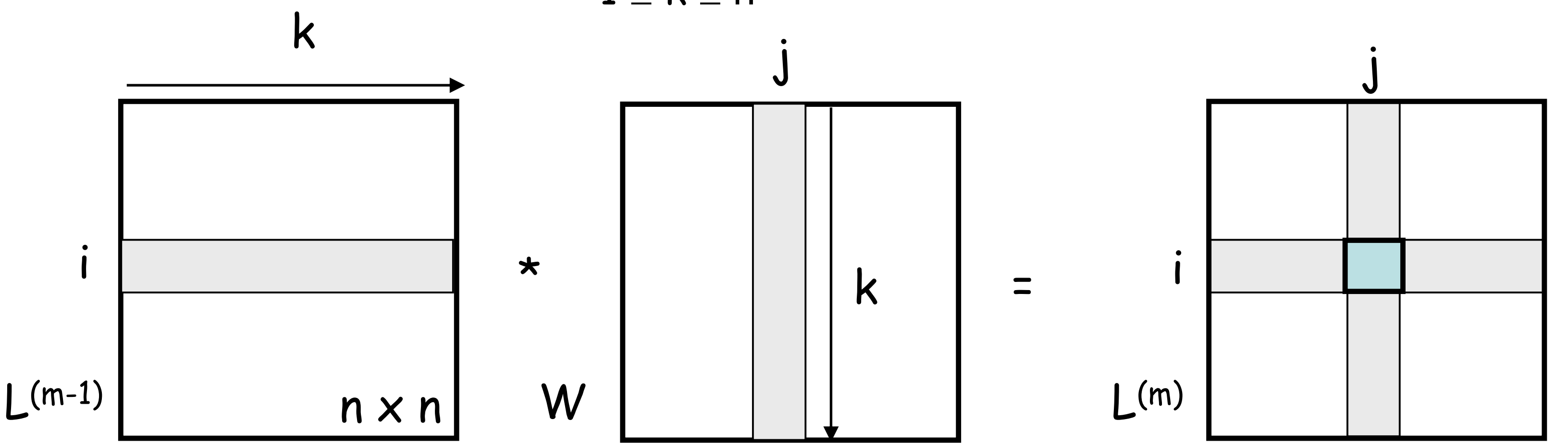
  $L^{(m)} = (l_{ij}^{(m)})$
- $L^{(n-1)}$ contains the actual shortest-path weights

  Given $L^{(m-1)}$ and $W \Rightarrow$ compute $L^{(m)}$
  - Extend the shortest paths computed so far by one more edge
- If the graph has no negative cycles: all simple shortest paths contain at most n - 1 edges

$$\delta(i, j) = l_{ij}^{(n-1)} \text{ and } l_{ij}^{(n)} = l_{ij}^{(n+1)} \ldots = l_{ij}^{(n-1)}$$

# Extending the Shortest Path

$$l_{ij}^{(m)} = \min_{1 \le k \le n} \{l_{ik}^{(m-1)} + w_{kj}\}$$




Replace: min $\rightarrow$ +

\+ $\rightarrow$ •

Computing $L^{(m)}$ looks like matrix multiplication

# EXTEND($L$, $W$, $n$)

1. create L', an n × n matrix
2. **for** $i \leftarrow 1$ **to** $n$
3. **do for** $j \leftarrow 1$ **to** $n$
4. **do** $l_{ij}' \leftarrow \infty$
5. **for** $k \leftarrow 1$ **to** $n$
6. **do** $l_{ij}' \leftarrow \min(l_{ij}', l_{ik} + w_{kj})$
7. **return** L'

$$l_{ij}^{(m)} = \min_{1 \le k \le n} \{l_{ik}^{(m-1)} + w_{kj}\}$$

Running time: $\Theta(n^3)$

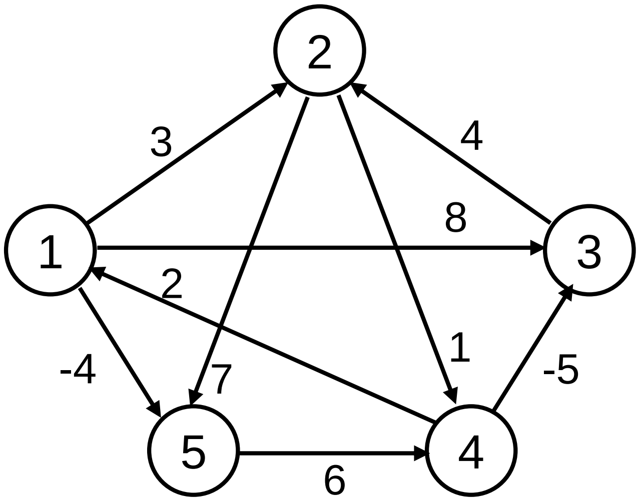
# SLOW-ALL-PAIRS-SHORTEST-PATHS($W$, $n$)

1. $L^{(1)} \leftarrow W$
2. **for** m ← 2 **to** n - 1
3. **do** $L^{(m)} \leftarrow$ EXTEND ($L^{(m-1)}$, $W$, $n$)
4. **return** $L^{(n-1)}$

Running time: $\Theta(n^4)$

# Example

$$l_{ij}^{(m)} = \min_{1 \le k \le n} \{l_{ik}^{(m-1)} + w_{kj}\}$$




$L^{(m-1)} = L^{(1)}$

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 8 | ∞ | -4 |
| ∞ | 0 | ∞ | 1 | 7 |
| ∞ | 4 | 0 | ∞ | ∞ |
| 2 | ∞ | -5 | 0 | ∞ |
| ∞ | ∞ | ∞ | 6 | 0 |

W

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 8 | ∞ | -4 |
| ∞ | 0 | ∞ | 1 | 7 |
| ∞ | 4 | 0 | ∞ | ∞ |
| 2 | ∞ | -5 | 0 | ∞ |
| ∞ | ∞ | ∞ | 6 | 0 |

$L^{(m)} = L^{(2)}$

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 8 | 2 | -4 |
| 3 | 0 | -4 | 1 | 7 |
| ∞ | 4 | 0 | 5 | 11 |
| 2 | -1 | -5 | 0 | -2 |
| 8 | ∞ | 1 | 6 | 0 |

… and so on until $L^{(4)}$

# Improving Running Time

- No need to compute all $L^{(m)}$ matrices
- If no negative-weight cycles exist:

$$L^{(m)} = L^{(n-1)} \text{ for all } m \geq n - 1$$

- We can compute $L^{(n-1)}$ by computing the sequence:

$L^{(1)} = W$

$L^{(2)} = W^2 = W \bullet W$

$L^{(4)} = W^4 = W^2 \bullet W^2$

$L^{(8)} = W^8 = W^4 \bullet W^4$ …

$$\Rightarrow 2^x = n - 1$$

$$L^{(n-1)} = W^{2^{\lceil \lg(n-1) \rceil}}$$

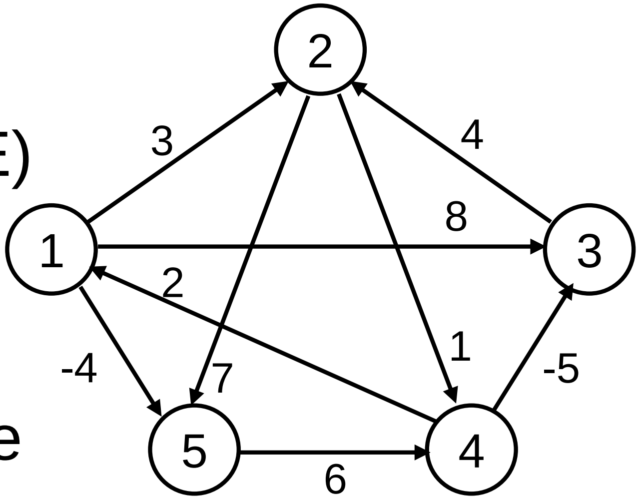
# FASTER-APSP(*W*, *n*)

```
L^(1) ← W
m ← 1
while m < n - 1
        do L^(2m) ← EXTEND(L^(m), L^(m), n)
           m ← 2*m
return L^(m)
```

- OK to overshoot: products don't change after $L^{(n-1)}$
- **Running Time:** $\Theta(n^3 \lg n)$

# The Floyd-Warshall Algorithm

- **Given:**
  - Directed, weighted graph G = (V, E)
  - Negative-weight edges may be present
  - No negative-weight cycles could be present in the graph
- **Compute:**
  - The shortest paths between all pairs of vertices in a graph



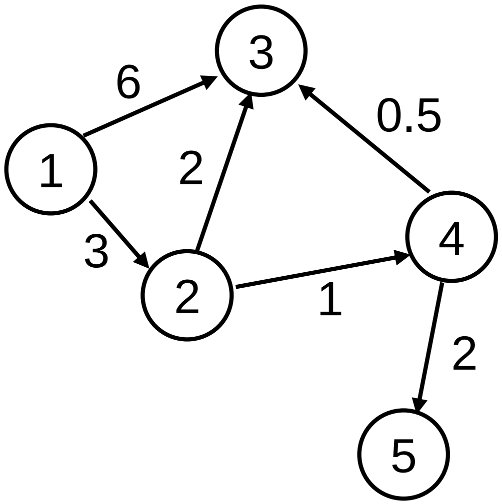
# The Structure of a Shortest Path

- Vertices in G are given by

  $V = \{1, 2, \ldots, n\}$

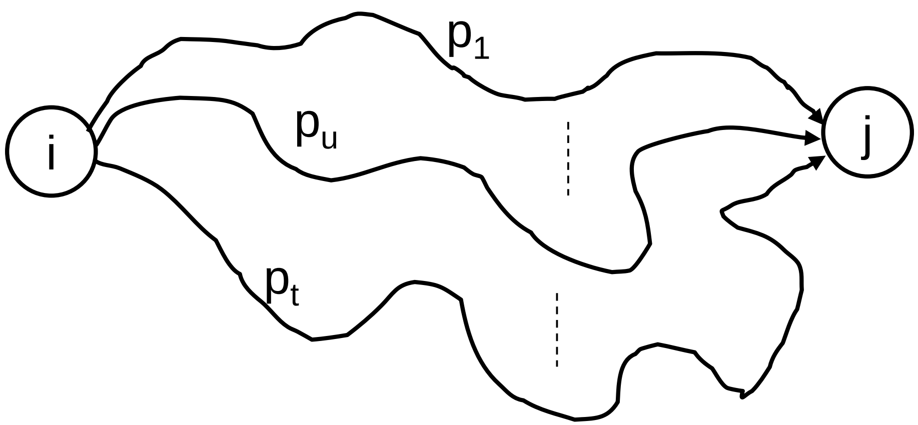
- Consider a path $p = \langle v_1, v_2, \ldots, v_l \rangle$
  - An **intermediate** vertex of $p$ is any vertex in the set $\{v_2, v_3, \ldots, v_{l-1}\}$
  - *E.g.:* $p = \langle 1, 2, 4, 5 \rangle$: $\{2, 4\}$

    $p = \langle 2, 4, 5 \rangle$: $\{4\}$



# The Structure of a Shortest Path

- For any pair of vertices $i, j \in V$, consider all paths from i to j whose intermediate vertices are all drawn from a subset {1, 2, ..., k}
  - Find p, a minimum-weight path from these paths

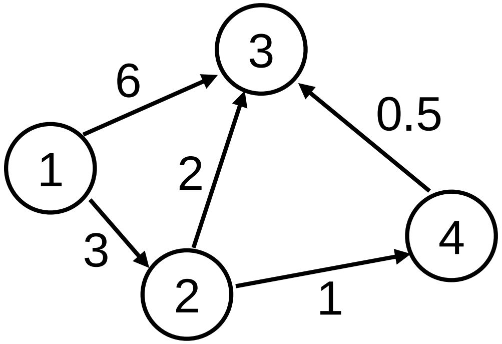



No vertex on these paths has index > k

# Example

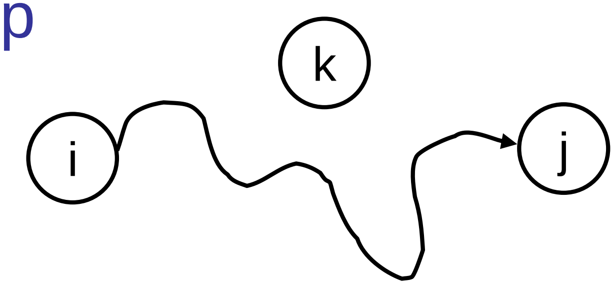
$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from {1, 2, ..., k}

- $d_{13}^{(0)} =$ 6
- $d_{13}^{(1)} =$ 6
- $d_{13}^{(2)} =$ 5
- $d_{13}^{(3)} =$ 5
- $d_{13}^{(4)} =$ 4.5

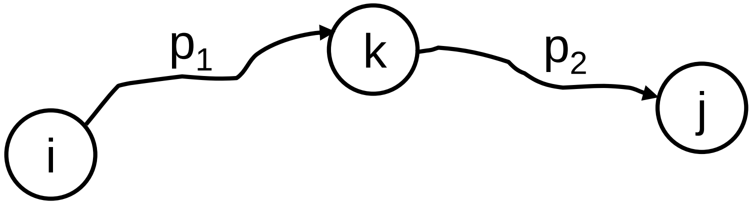


# The Structure of a Shortest Path

- k is not an intermediate vertex of path p
  - Shortest path from i to j with intermediate vertices from {1, 2, …, k} is a shortest path from i to j with intermediate vertices from {1, 2, …, k - 1}




- k is an intermediate vertex of path p
  - $p_1$ is a shortest path from i to k
  - $p_2$ is a shortest path from k to j
  - k is not intermediary vertex of $p_1$, $p_2$
  - $p_1$ and $p_2$ are shortest paths from i to k with vertices from {1, 2, …, k - 1}



# A Recursive Solution (cont.)

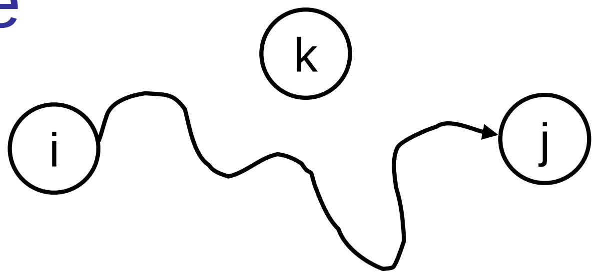
$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from $\{1, 2, \ldots, k\}$

- $k = 0$
- $d_{ij}^{(k)} = w_{ij}$

# A Recursive Solution (cont.)

$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from $\{1, 2, \ldots, k\}$

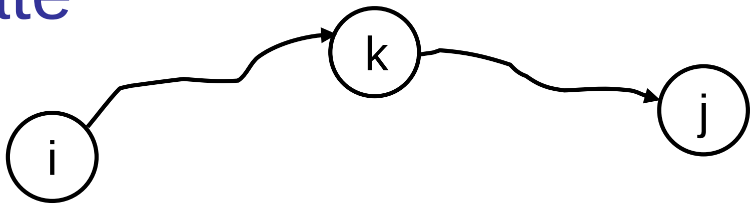
- $k \geq 1$
- **Case 1:** k is not an intermediate vertex of path p
- $d_{ij}^{(k)} = d_{ij}^{(k-1)}$



# A Recursive Solution (cont.)

$d_{ij}^{(k)}$ = the weight of a shortest path from vertex i to vertex j with all intermediary vertices drawn from $\{1, 2, \ldots, k\}$

- $k \geq 1$
- **Case 2:** k is an intermediate vertex of path p
- $d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

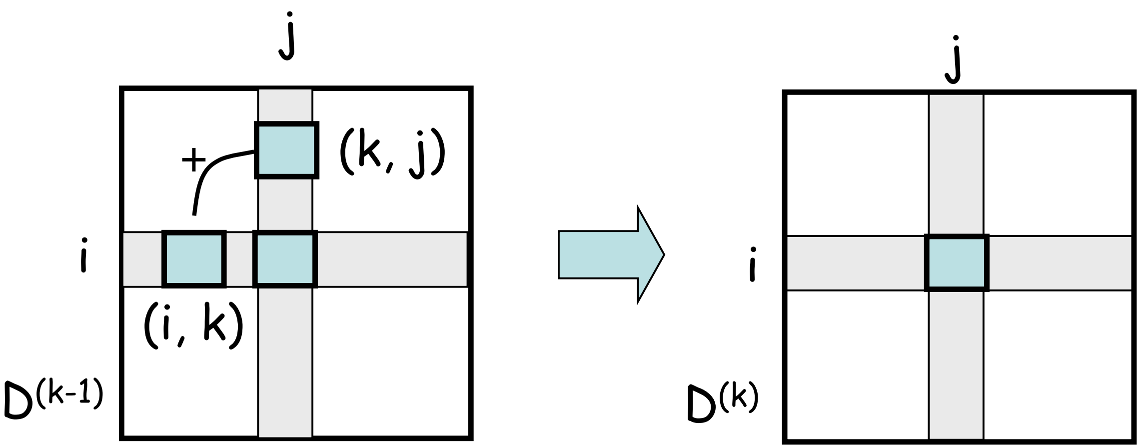


# Computing the Shortest Path Weights

- $d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} & \text{if } k \geq 1 \end{cases}$

- The final solution: $D^{(n)} = (d_{ij}^{(n)})$:

  $d_{ij}^{(n)} = \delta(i, j)\ \forall\ i, j \in V$



# The Floyd-Warshall algorithm

```
Floyd-Warshall(W[1..n][1..n])
D ← W    // D^(0)
for k ← 1 to n do // compute D^(k)
   for i ←1 to n do
      for j ←1 to n do
        if D[i][k] + D[k][j] < D[i][j] then
           D[i][j] ← D[i][k] + D[k][j]
return D
```

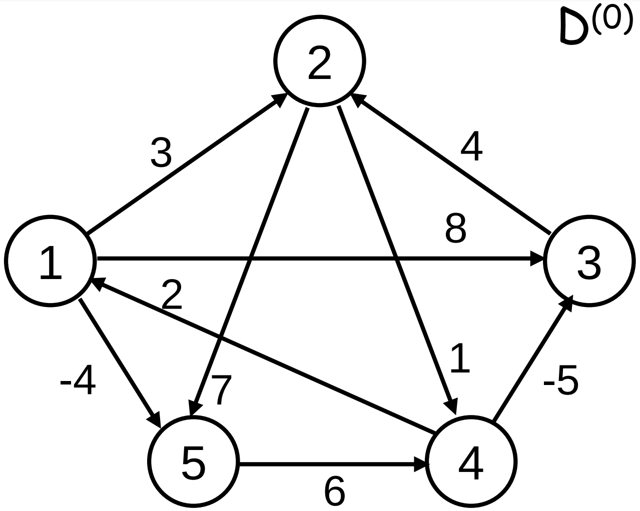
**Running Time: $O(n^3)$**

# Computing predecessor matrix

- *How do we compute the predecessor matrix?*
  - Initialization: $$p^{(0)}(i,j) = \begin{cases} nil & \text{if } i = j \text{ or } w_{ij} = \infty \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty \end{cases}$$
- – Updating: $p^{(k)}(i,j) = p^{(k-1)}(i,j)$ *if*$(d^{(k-1)}(i,j) <= d^{(k-1)}(i,k) + (d^{(k-1)}(k,j))$
- – $p^{(k-1)}(k,j)$ *if*$(d^{(k-1)}(i,j) > d^{(k-1)}(i,k) + (d^{(k-1)}(k,j))$

```
Floyd-Warshall(W[1..n][1..n])
…
for k ←1 to n do // compute D(k)
    for i ←1 to n do
        for i ←1 to n do
            if D[i][k] + D[k][j] < D[i][j] then
                D[i][j] ←D[i][k] + D[k][j]
                P[i][j] ←P[k][j]
return D
```

# Example $d_{ij}^{(k)} = \min \{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$




$D^{(0)} = W$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 3 | 8 | ∞ | -4 |
| 2 | ∞ | 0 | ∞ | 1 | 7 |
| 3 | ∞ | 4 | 0 | ∞ | ∞ |
| 4 | 2 | ∞ | -5 | 0 | ∞ |
| 5 | ∞ | ∞ | ∞ | 6 | 0 |

$D^{(1)}$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 3 | 8 | ∞ | -4 |
| 2 | ∞ | 0 | ∞ | 1 | 7 |
| 3 | ∞ | 4 | 0 | ∞ | ∞ |
| 4 | 2 | 5 | -5 | 0 | -2 |
| 5 | ∞ | ∞ | ∞ | 6 | 0 |

$D^{(2)}$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 3 | 8 | 4 | -4 |
| 2 | ∞ | 0 | ∞ | 1 | 7 |
| 3 | ∞ | 4 | 0 | 5 | 11 |
| 4 | 2 | 5 | -5 | 0 | -2 |
| 5 | ∞ | ∞ | ∞ | 6 | 0 |

$D^{(3)}$

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 8 | 4 | -4 |
| ∞ | 0 | ∞ | 1 | 7 |
| ∞ | 4 | 0 | 5 | 11 |
| 2 | -1 | -5 | 0 | -2 |
| ∞ | ∞ | ∞ | 6 | 0 |

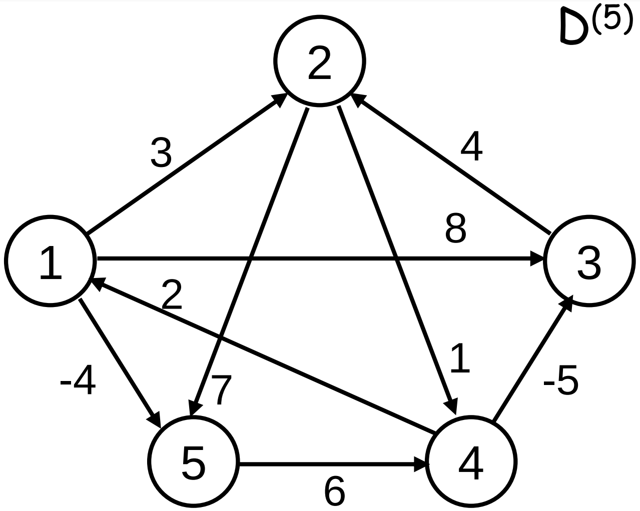
$D^{(4)}$

| | | | | |
|---|---|---|---|---|
| 0 | 3 | -1 | 4 | -4 |
| 3 | 0 | -4 | 1 | -1 |
| 7 | 4 | 0 | 5 | 3 |
| 2 | -1 | -5 | 0 | -2 |
| 8 | 5 | 1 | 6 | 0 |

------

# Example

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$




$D^{(5)}$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | -3 | 2 | -4 |
| 2 | 3 | 0 | -4 | 1 | -1 |
| 3 | 7 | 4 | 0 | 5 | 3 |
| 4 | 2 | -1 | -5 | 0 | -2 |
| 5 | 8 | 5 | 1 | 6 | 0 |

$P^{(5)}$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | - | 3 | 4 | 5 | 1 |
| 2 | 4 | - | 4 | 2 | 1 |
| 3 | 4 | 3 | - | 2 | 1 |
| 4 | 4 | 3 | 4 | - | 1 |
| 5 | 4 | 3 | 4 | 5 | - |

Source: 5, Destination: 1

Shortest path: 8

Path: 5 …1 : 5…4…1: 5->4…1: 5->4->1

Source: 1, Destination: 3

Shortest path: -3

Path: 1 …3 : 1…4…3: 1…5…4…3: 1->5->4->3

# PrintPath for Warshall’s Algorithm

```
PrintPath(s, t)
{
   if(P[s][t]==nil) {print(“No path”);return;}
   else if (P[s][t]==s){
        print(s);
   }
   else{
        print_path(s,P[s][t]);
        print_path(P[s][t], t);
   }
}
Print (t) at the end of the PrintPath(s,t)
```

# Question

- Why should we use D[i, j] instead of $D^{(k)}$[i, j]?
- Exercise:
  - 25.2-4: Memory $O(n^2)$
  - 25.2-6: Negative weight cycle
  - Find the shortest positive cycle

# Transitive closure of the graph

- Input:
  - Un-weighted graph $G$: $W[i][j] = 1$, if $(i,j) \in E$, $W[i][j] = 0$ otherwise.
- Output:
  - $T[i][j] = 1$, if there is a path from $i$ to $j$ in $G$, $T[i][j] = 0$ otherwise.
- Algorithm:
  - Just run Floyd-Warshall with weights 1, and make $T[i][j] = 1$, whenever $D[i][j] < \infty$.
  - More efficient: use only Boolean operators

# Transitive closure algorithm

```
Transitive-Closure(W[1..n][1..n])
T ← W    // T^(0)
for k ← 1 to n do // compute T^(k)
    for i ←1 to n do
        for i ←1 to n do
            T[i][j] ← T[i][j] ∨ (T[i][k] ∧ T[k][j])
return T
```

# Readings

- Chapters 25