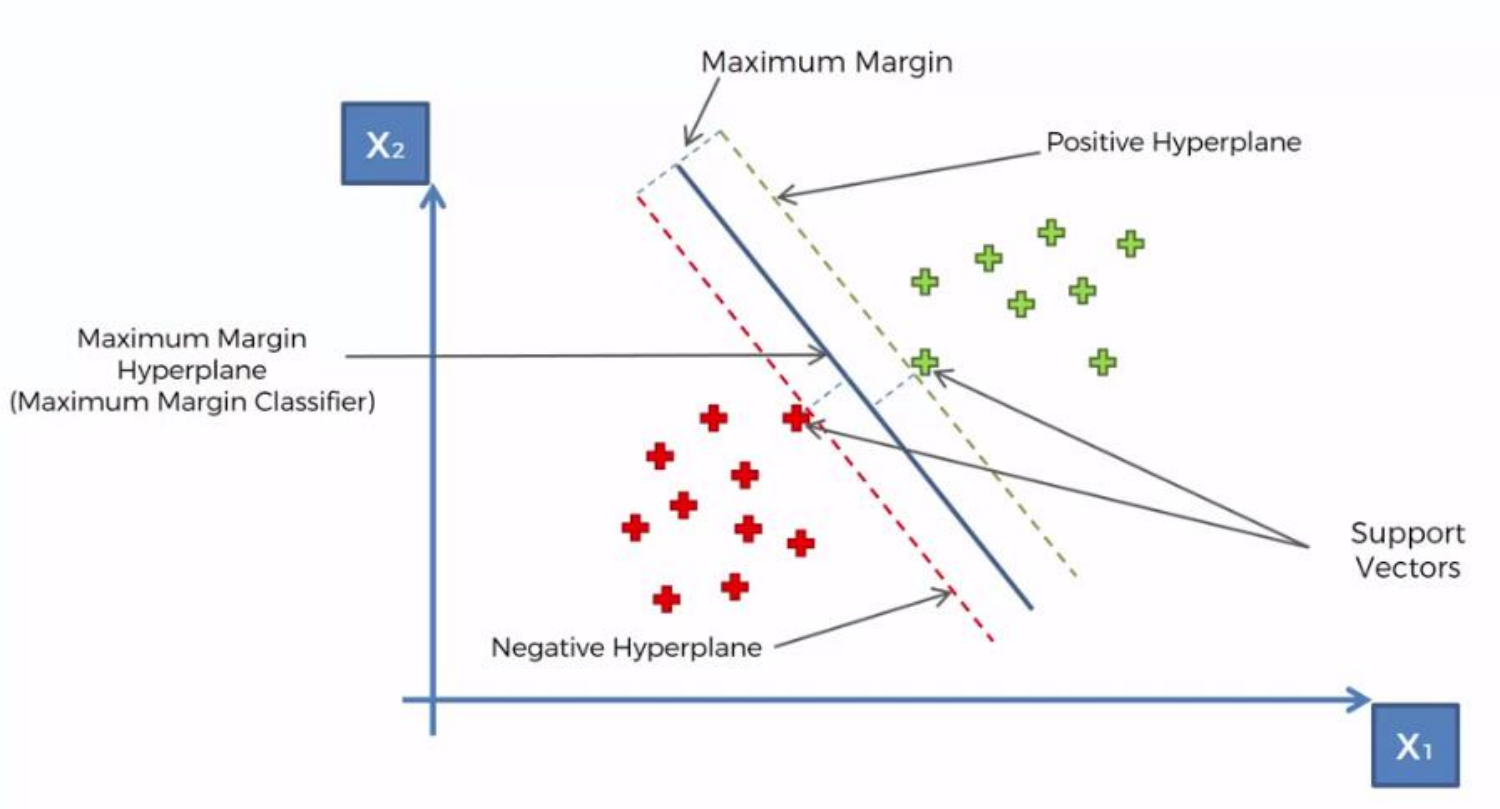
SUPPORT VECTOR MACHINES (SVM)

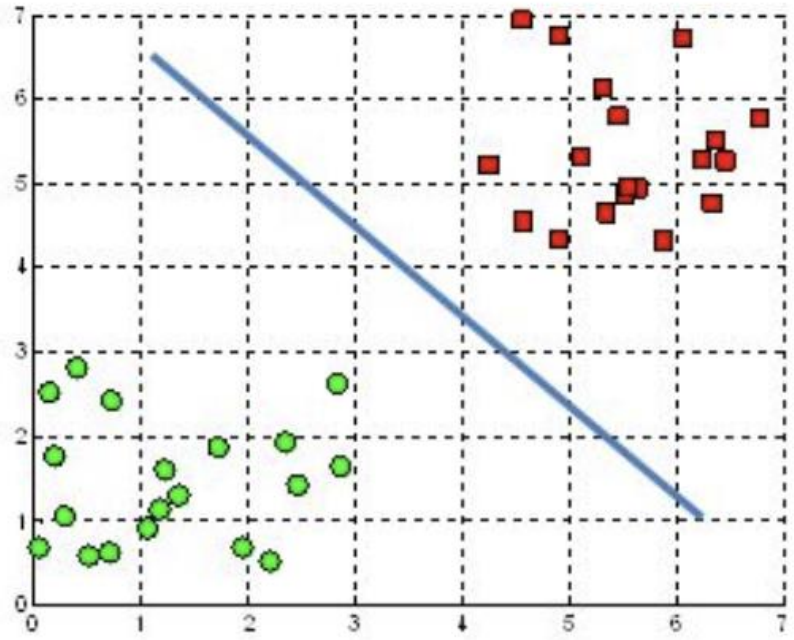
Support vector machines (SVM)

- **Support Vector Machines (SVM)** are supervised learning models used for classification and regression analysis.
- SVMs are based on the idea of finding a **hyperplane** that best divides a dataset into two classes

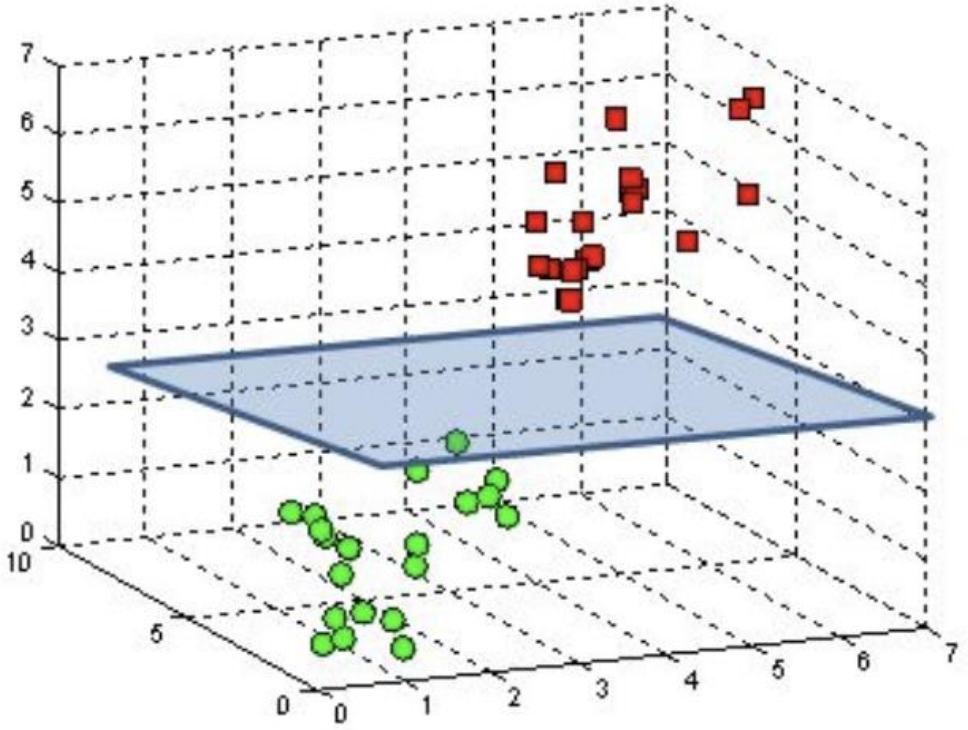


What is a hyperplane?

A hyperplane in $\mathbb{R}^2$ is a line



A hyperplane in $\mathbb{R}^3$ is a plane



How do we find the right hyperplane?

The goal is to choose a hyperplane with the greatest possible margin between the hyperplane

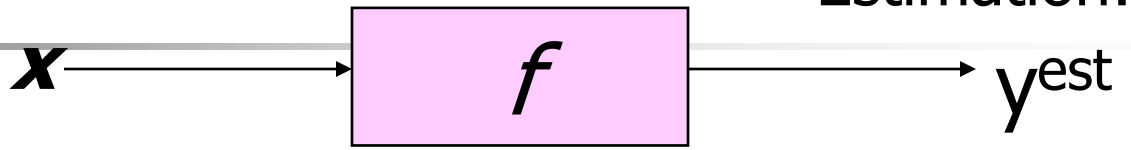
How can we identify the right hyper-plane?

The goal is to choose a hyperplane with the greatest possible margin between the hyperplane

"Select the hyper-plane which segregates the two classes better".

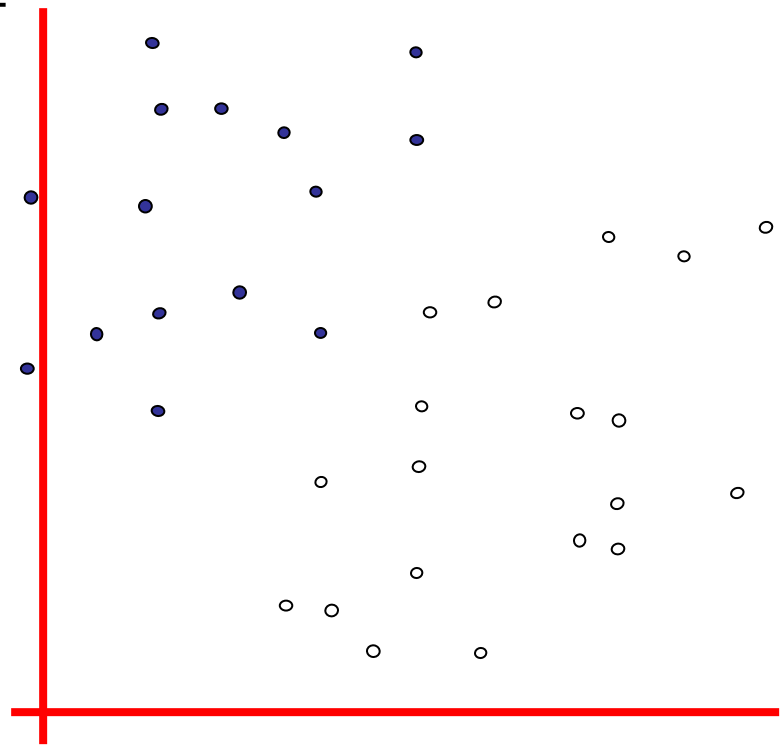
Linear Classifiers

Estimation:



$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} - b)$

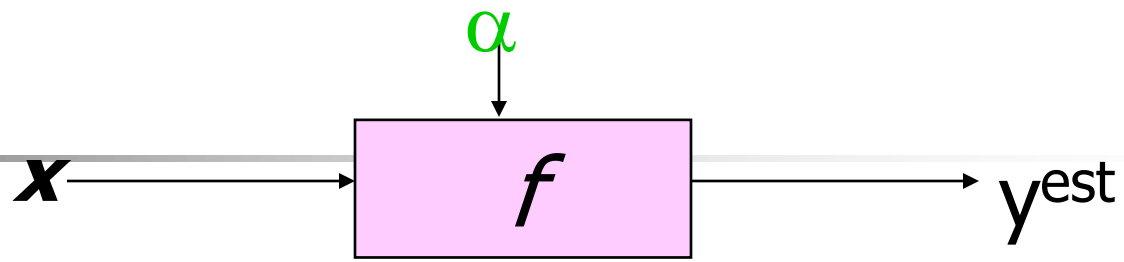
- denotes +1
- denotes -1



w: weight vector
x: data vector

How would you classify this data?

Linear Classifiers



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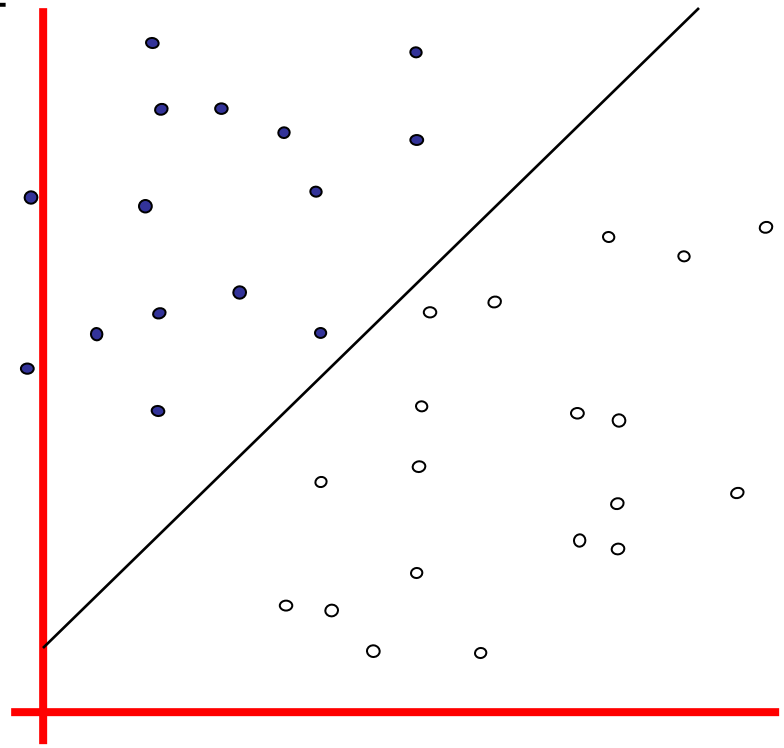
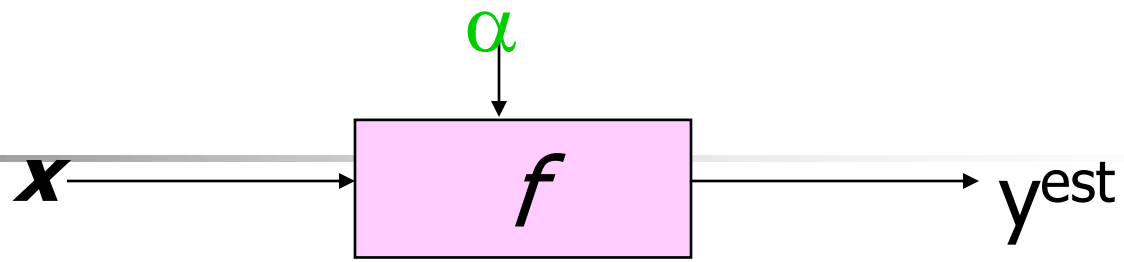


How would you classify this data?

Linear Classifiers



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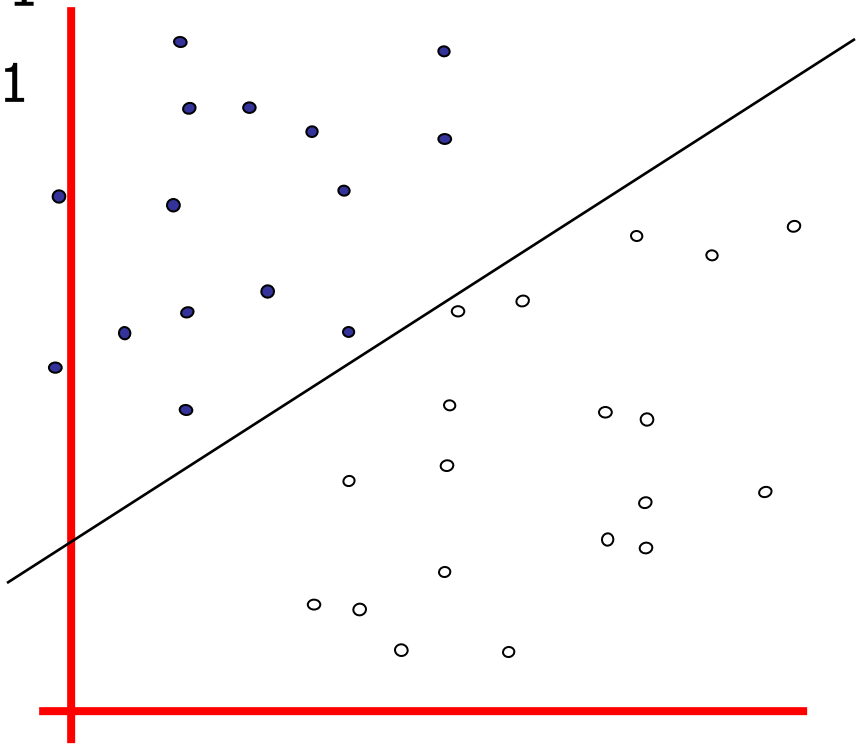
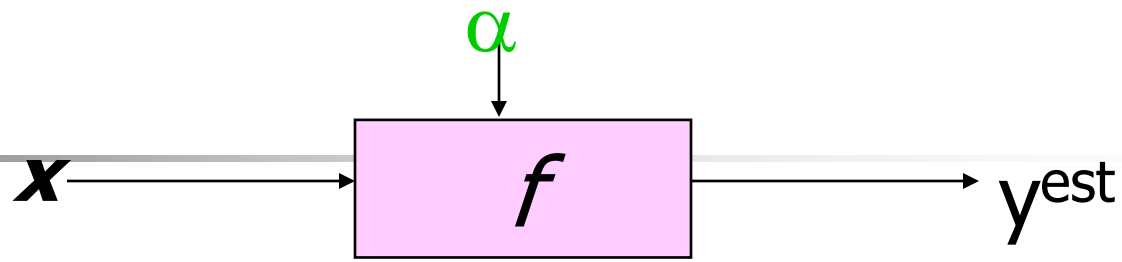


How would you classify this data?

Linear Classifiers



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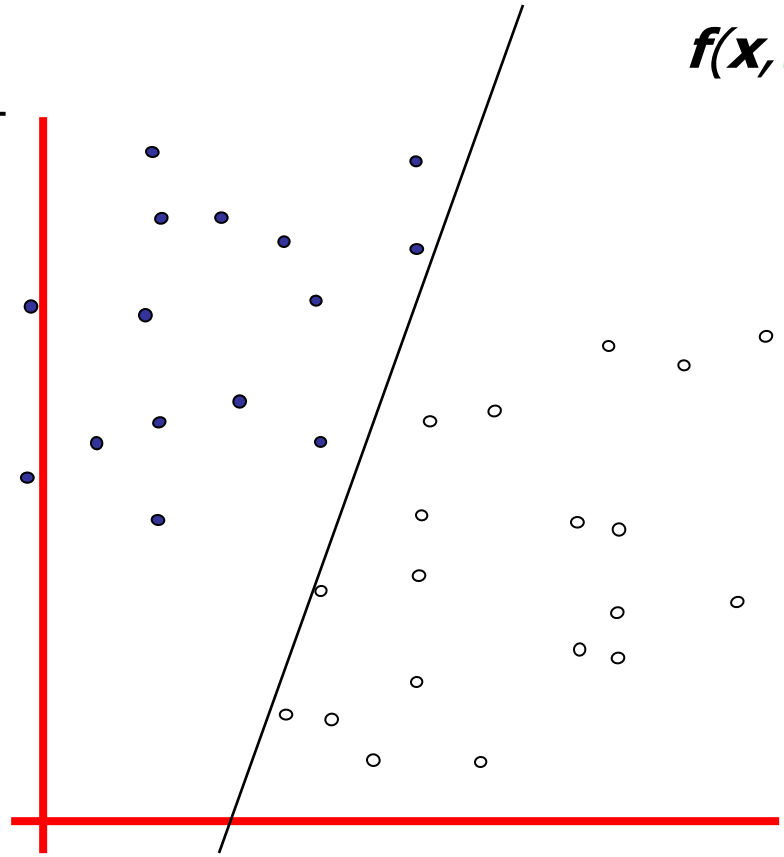
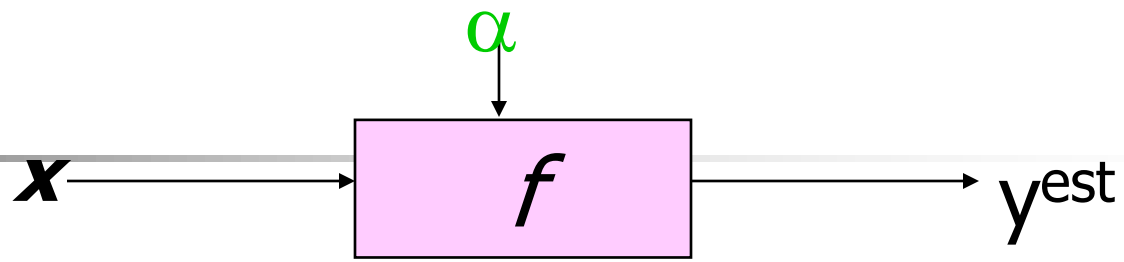


How would you classify this data?

Linear Classifiers



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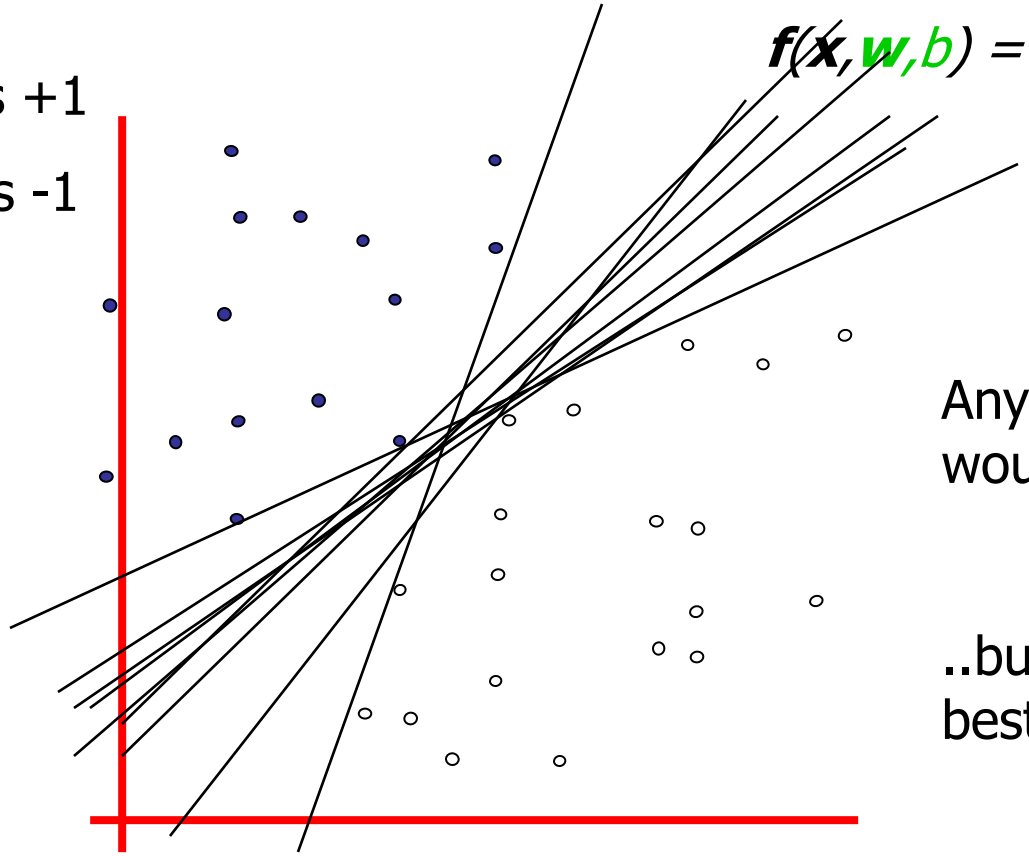
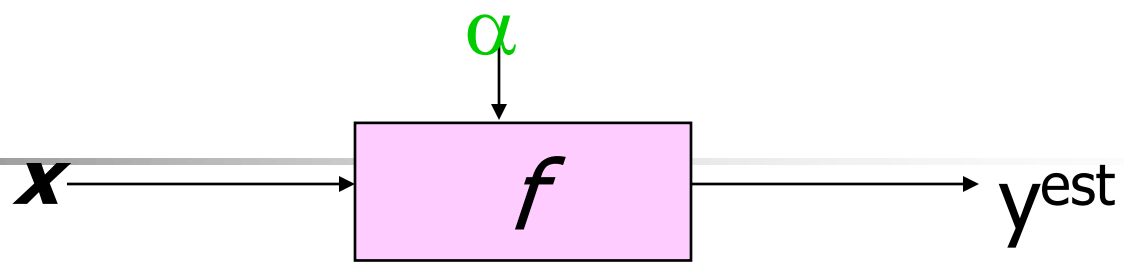


Any of these would be fine..

..but which is best?

Classifier Margin



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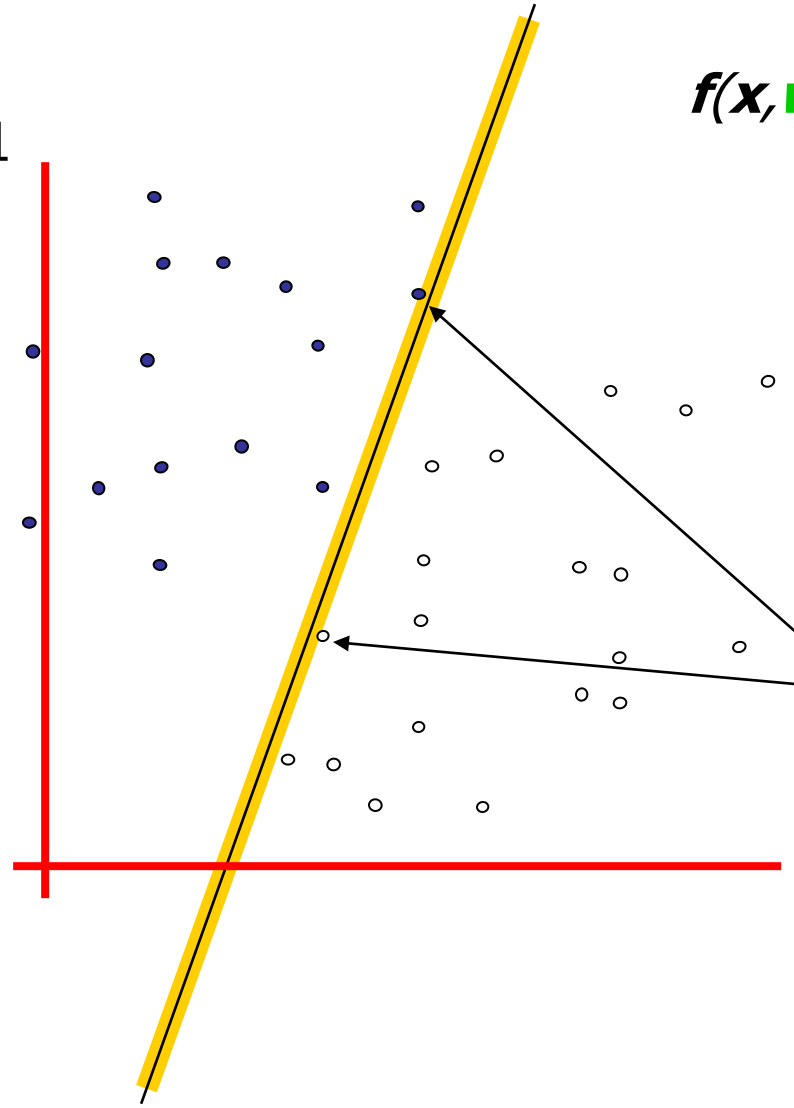
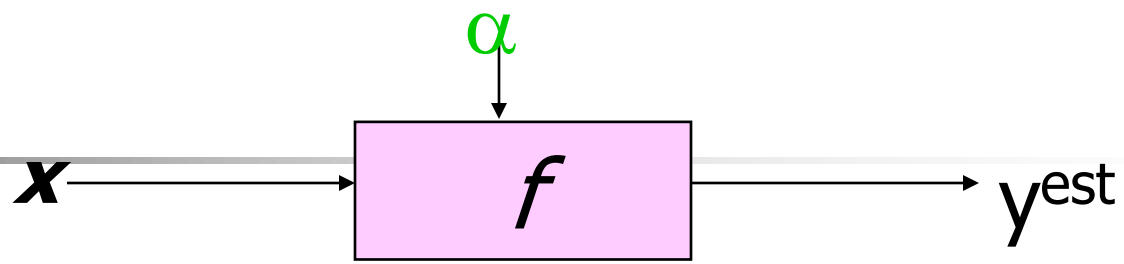


Define the margin of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.

Maximum Margin



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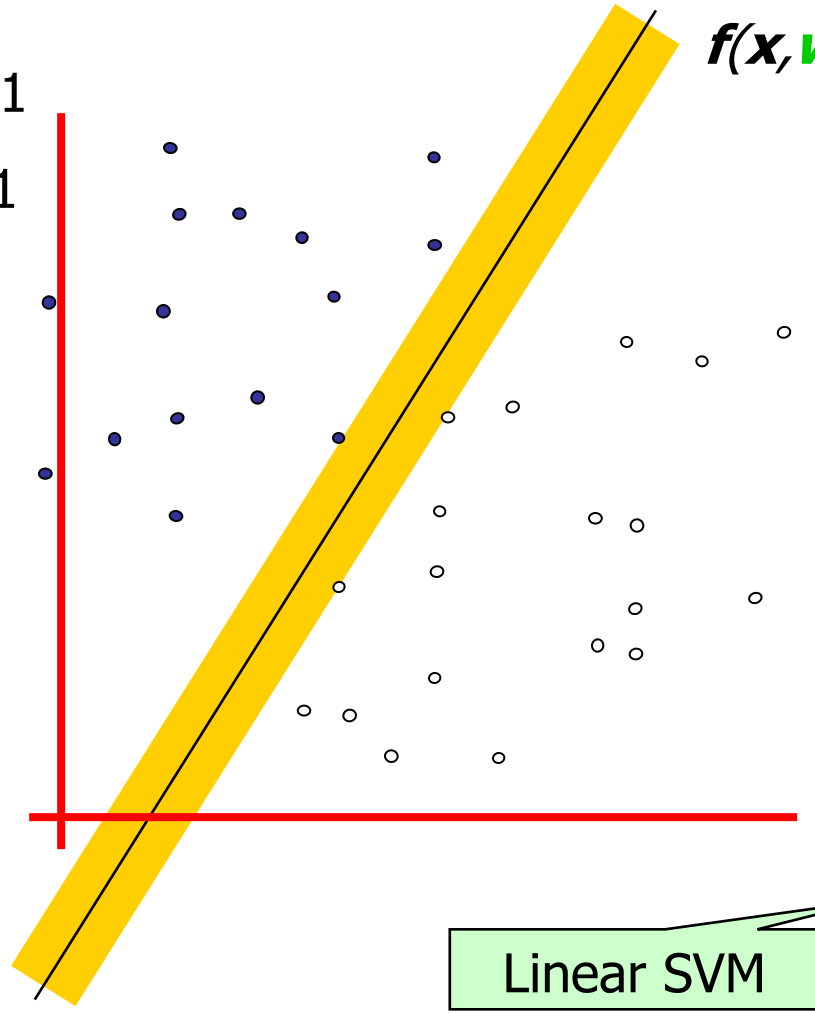
- denotes +1
- denotes -1

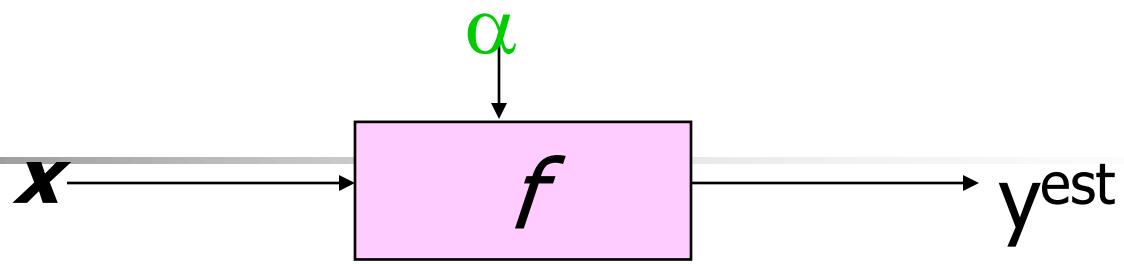


The maximum margin linear classifier is the linear classifier with the, um, maximum margin.

This is the simplest kind of SVM (Called an LSVM)

Linear SVM

Maximum Margin



$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\mathbf{x} + b)$

- denotes +1
- denotes -1

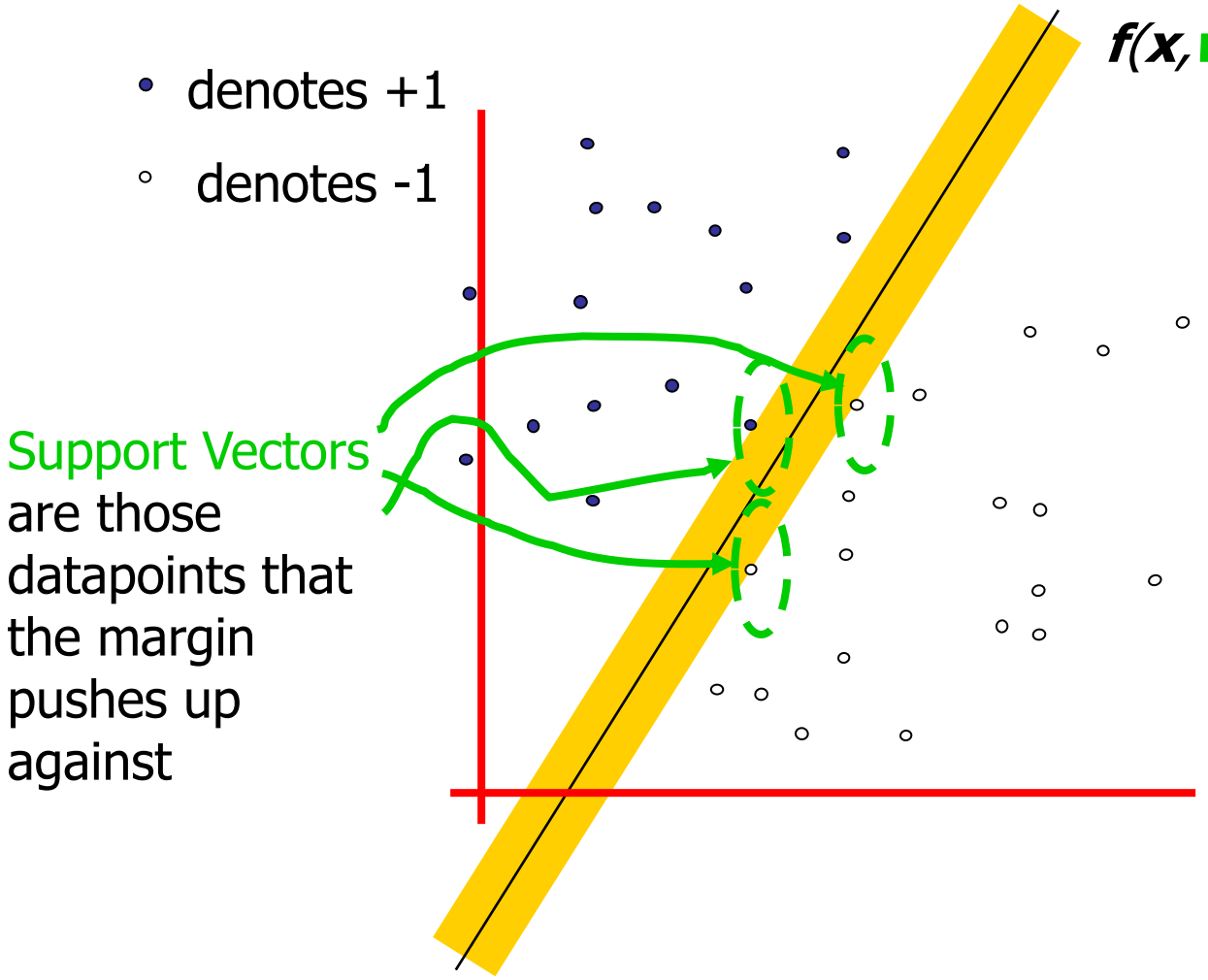
Support Vectors are those datapoints that the margin pushes up against

The maximum margin linear classifier is the linear classifier with the, um, maximum margin.

This is the simplest kind of SVM (Called an LSVM)

Linear SVM

Why Maximum Margin?

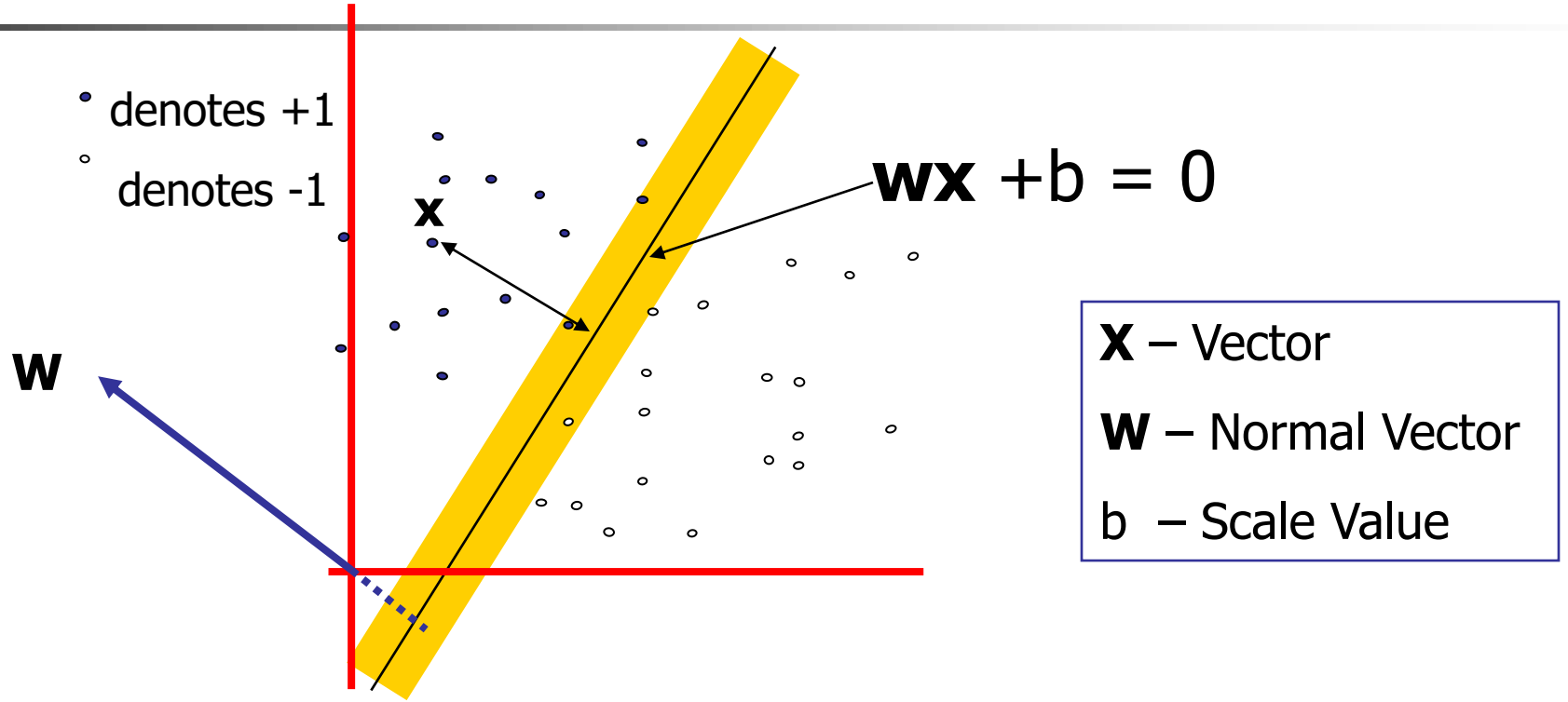


$f(\mathbf{x}, \mathbf{w}, b) = sign(\mathbf{w}.\ \mathbf{x} - b)$

The maximum margin linear classifier is the linear classifier with the maximum margin.

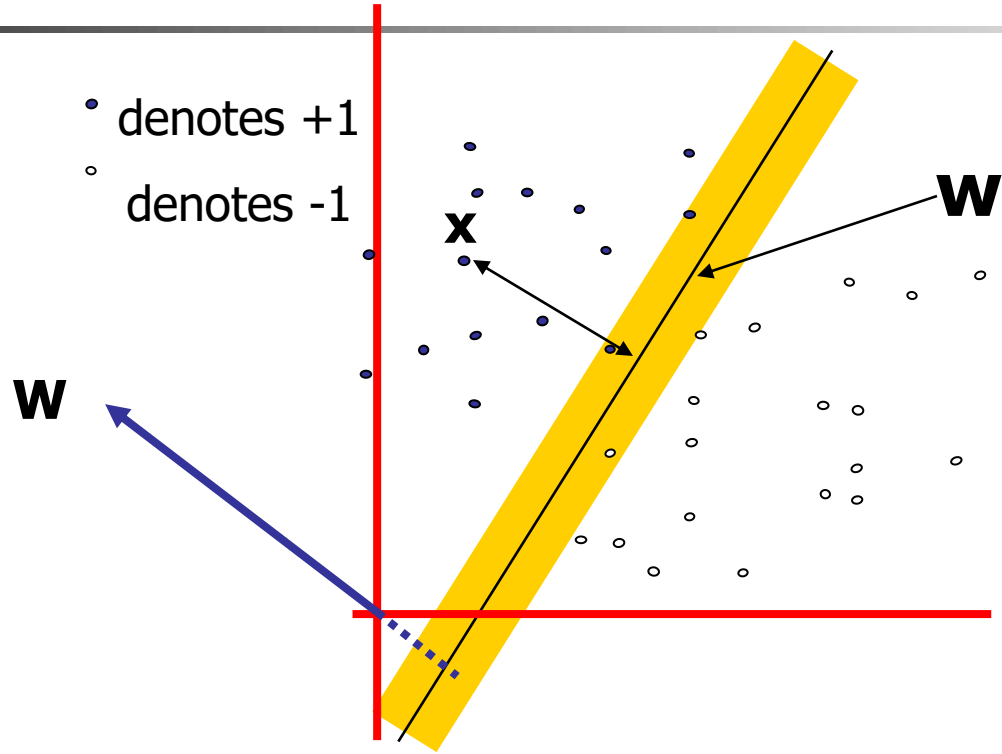
This is the simplest kind of SVM (Called an LSVM)

How to calculate the distance from a point to a line?



- http://mathworld.wolfram.com/Point-LineDistance2-Dimensional.html
- In our case, $w_1*x_1+w_2*x_2+b=0$,
- thus, $\mathbf{w}=(w_1,w_2)$, $\mathbf{x}=(x_1,x_2)$

Estimate the Margin



x – Vector

w – Normal Vector

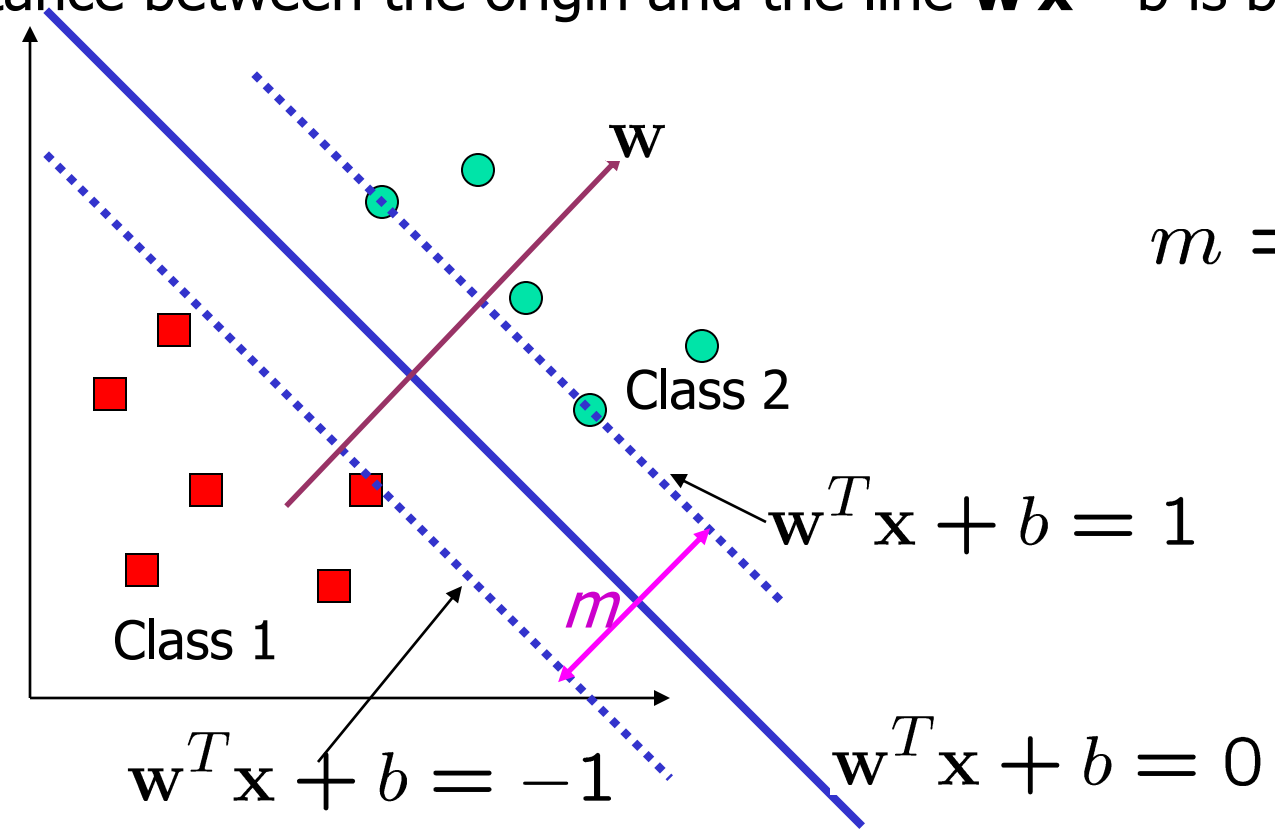
b – Scale Value

- What is the distance expression for a point **x** to a line **wx**+b= 0?

$$d(\mathbf{x}) = \frac{\left|\mathbf{x}\cdot\mathbf{w}+b\right|}{\sqrt{\left\|\mathbf{w}\right\|_2^2}} = \frac{\left|\mathbf{x}\cdot\mathbf{w}+b\right|}{\sqrt{\sum_{i=1}^{d} w_i^2}}$$

Large-margin Decision Boundary

- The decision boundary should be as far away from the data of both classes as possible
  - We should maximize the margin, *m*
  - Distance between the origin and the line $\mathbf{w}^t\mathbf{x}=-b$ is $b/||\mathbf{w}||$



$$m = \frac{2}{||\mathbf{w}||}$$

Finding the Decision Boundary

- Let $\{x_1, ..., x_n\}$ be our data set and let $y_i \in \{1,-1\}$ be the class label of x_i
- The decision boundary should classify all points correctly $\Rightarrow$

$$y_i(\mathbf{w}^T\mathbf{x}_i + b) \geq 1, \qquad \forall i$$

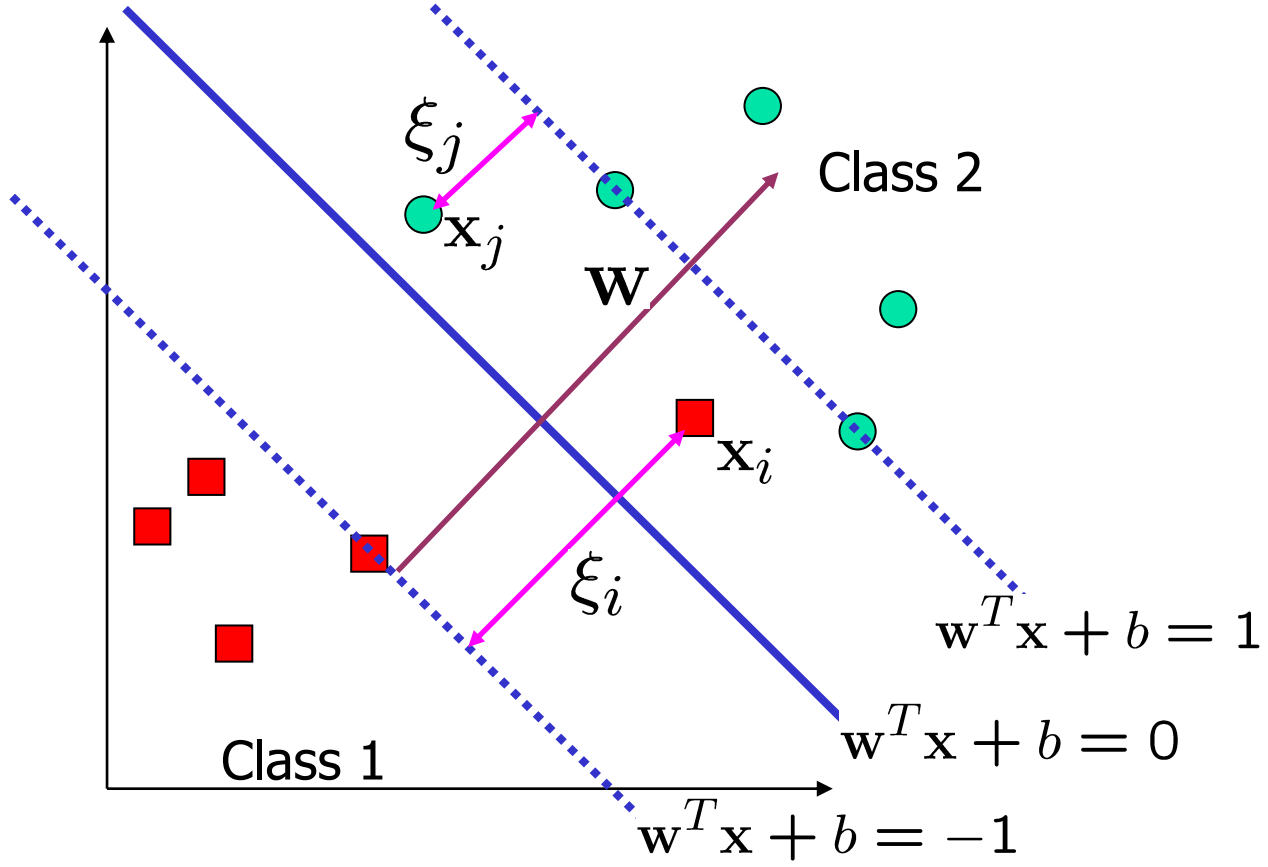
- To see this: when y=-1, we wish (wx+b)<1, when y=1, we wish (wx+b)>1. For support vectors, we wish y(wx+b)=1.
- The decision boundary can be found by solving the following constrained optimization problem

$$\text{Minimize } \frac{1}{2}||\mathbf{w}||^2$$

$$\text{subject to } y_i(\mathbf{w}^T\mathbf{x}_i + b) \geq 1 \qquad \forall i$$

Allowing errors in our solutions

- We allow "error" ξ_i in classification; it is based on the output of the discriminant function $\mathbf{w}^T\mathbf{x}+b$
- ξ_i approximates the number of misclassified samples



Soft Margin Hyperplane

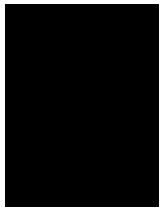
- If we minimize $\sum_i \xi_i$, ξ_i can be computed by

$$\begin{cases} \mathbf{w}^T\mathbf{x}_i + b \geq 1 - \xi_i & y_i = 1 \\ \mathbf{w}^T\mathbf{x}_i + b \leq -1 + \xi_i & y_i = -1 \\ \xi_i \geq 0 & \forall i \end{cases}$$

  - ξ_i are "slack variables" in optimization
  - Note that ξ_i=0 if there is no error for $\mathbf{x}_i$
  - ξ_i is an upper bound of the number of errors
- We want to minimize $\frac{1}{2}||\mathbf{w}||^2 + C\sum_{i=1}^{n}\xi_i$
  - C: tradeoff parameter between error and margin
- The optimization problem becomes

$$\text{Minimize } \frac{1}{2}||\mathbf{w}||^2 + C\sum_{i=1}^{n}\xi_i$$
$$\text{subject to } y_i(\mathbf{w}^T\mathbf{x}_i + b) \geq 1 - \xi_i, \quad \xi_i \geq 0$$

The **total error** of a SVM, is the sum of the classification error and the margin error.

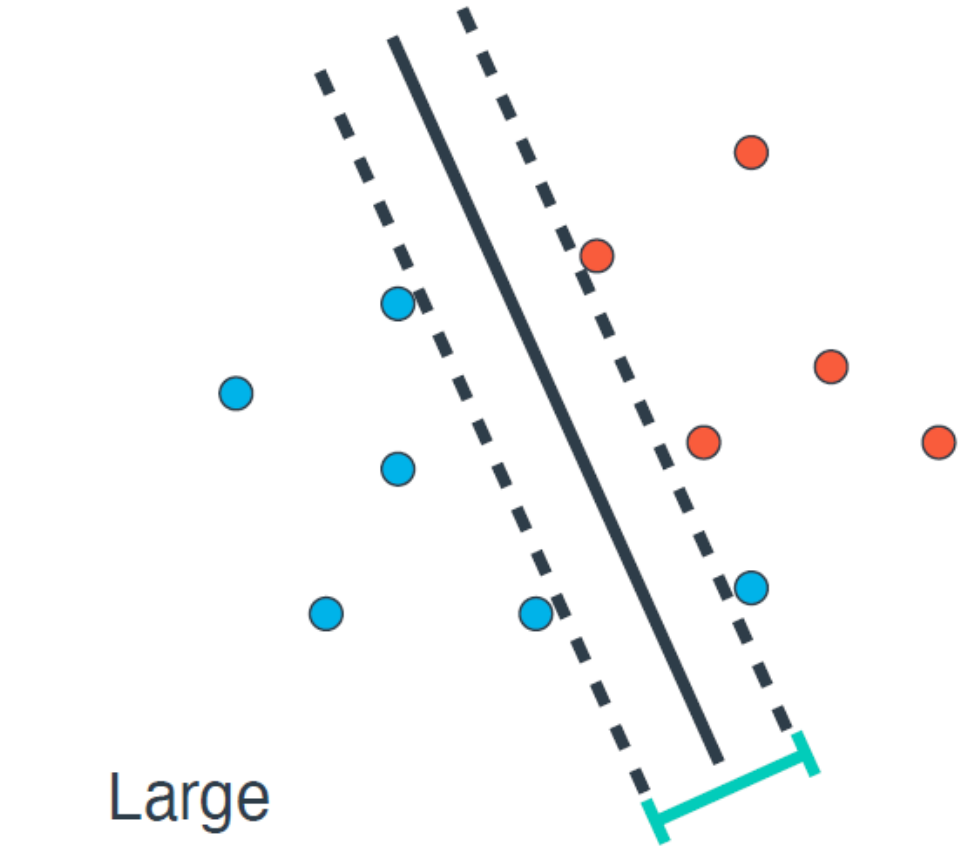


Error function(s)

1. When a misclassification occurs, it is because a given point is on the wrong side of the separating hyperplane, and that's called a **classification error**.

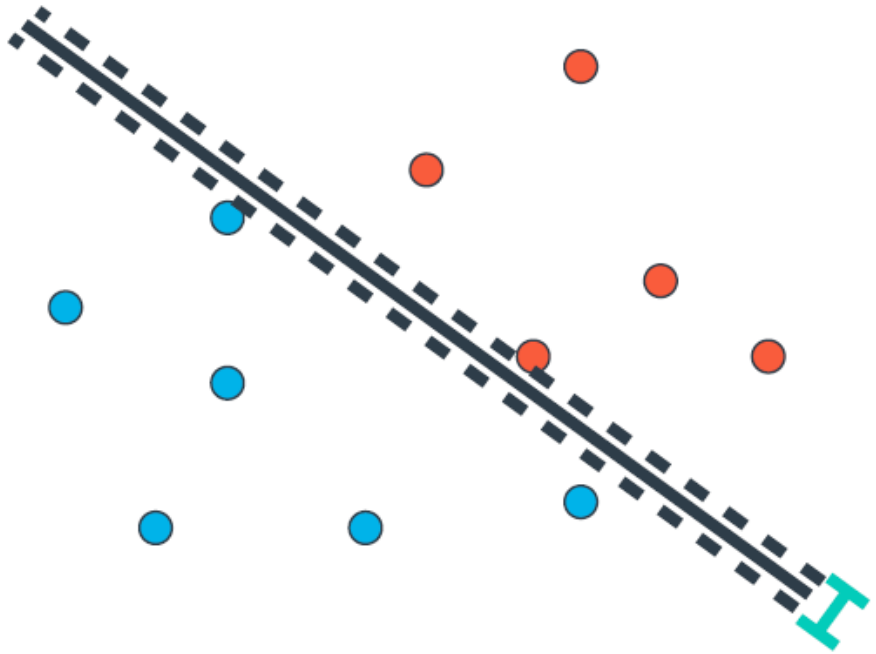
2. Whenever a point is inside the margin, that counts as a **margin error**.

Margin Error



Large classification error

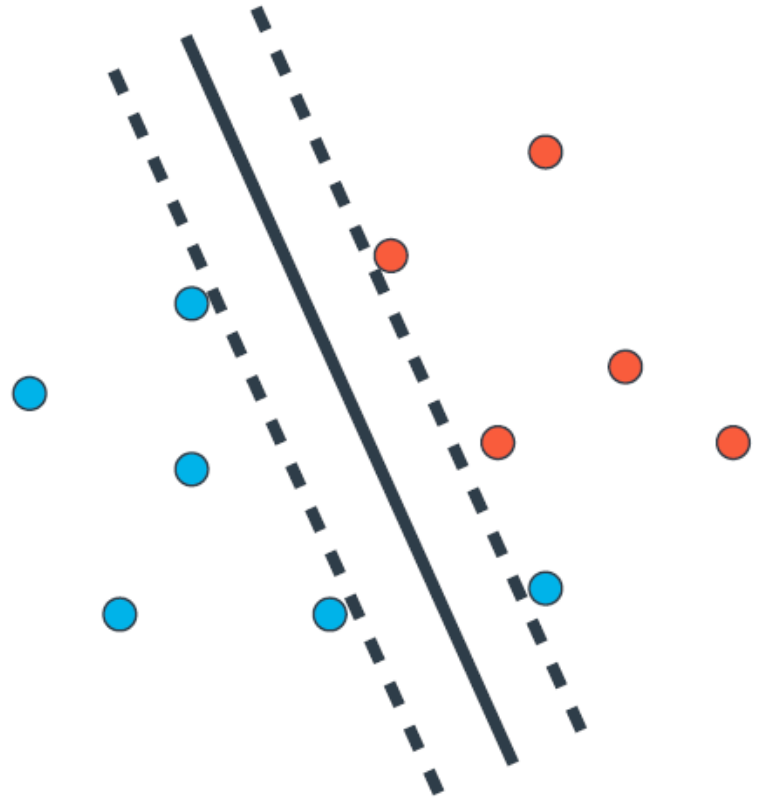
Large margin
Small margin error



Small classification error

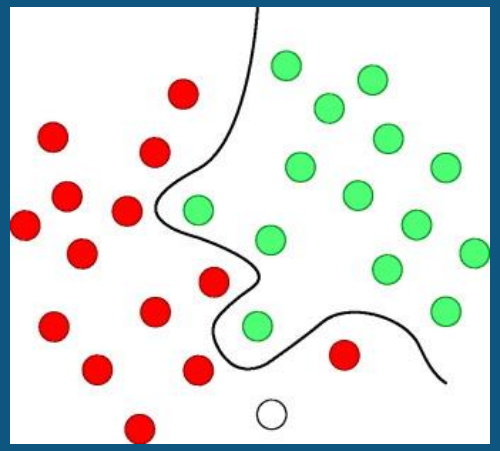
Small margin
Large margin error

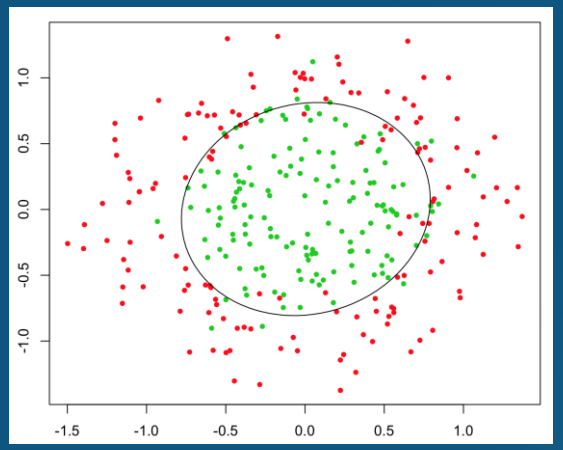
The C parameter



C Classification Error + Margin Error

Support Vector machine (SVM)



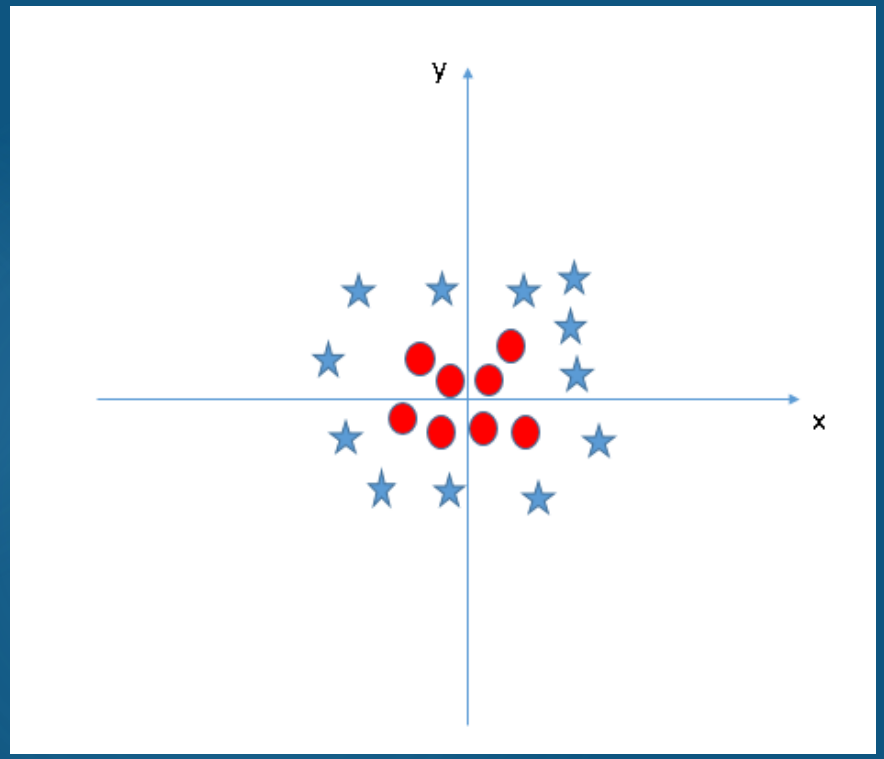


Support Vector Machines:

When non-linear boundary is used for the classification purpose, it is called Support Vector Machines (SVMs). It is an extension of the Support Vector Classifier and done by enlarging the feature space in a specific way, using *kernels*.

Find the hyperplane

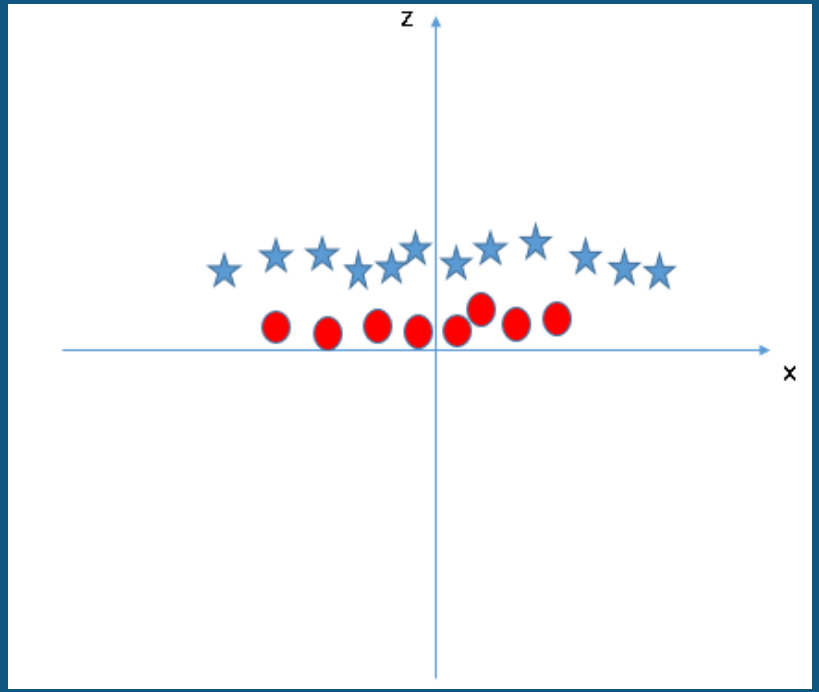
In the scenario below, we can't have linear hyperplane between the two classes, so how does SVM classify these two classes? Till now, we have only looked at the linear hyperplane.



SVM can solve this problem. It solves this problem by introducing additional feature. Here, we will add a new feature $z=x^2+y^2$.

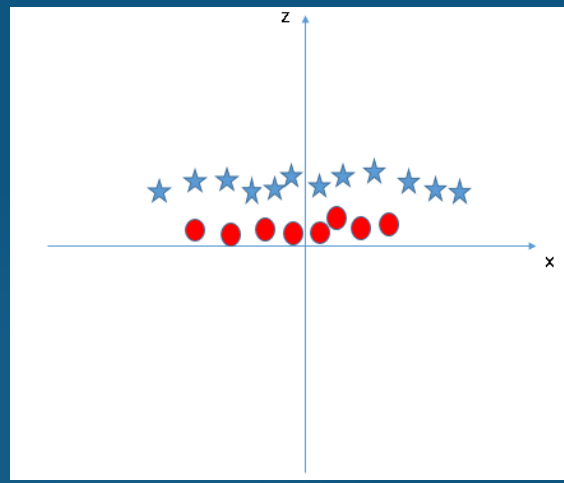
Find the hyperplane

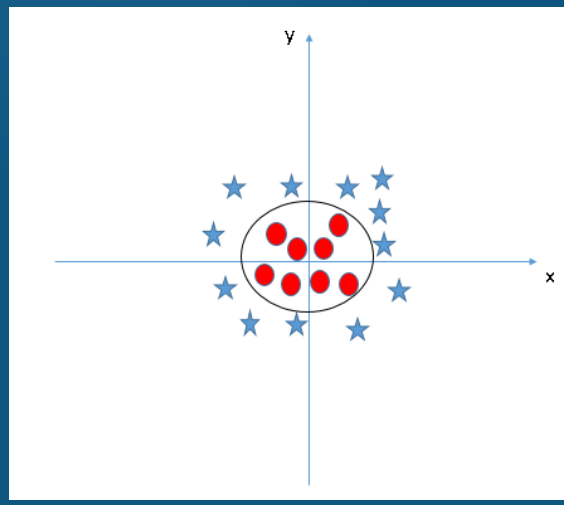
- Now, let's plot the data points on axis x and z:
- In this plot, points to consider are:
  - All values for z would be positive always because z is the squared sum of both x and y
  - In the original plot, red circles appear close to the origin of x and y axes, leading to lower value of z and star relatively away from the origin result to higher value of z.



KERNEL TRICK

- In SVM, it is easy to have a linear hyperplane between these two classes. But, another burning question which arises is, should we need to add this feature manually to have a hyperplane.
- No, SVM has a technique called the **kernel trick**. These are functions which takes low dimensional input space and transform it to a higher dimensional space i.e. it converts non separable problem to separable problem, these functions are called **kernels**.
- It is mostly useful in non-linear separation problem. Simply put, it does some extremely complex data transformations, then find out the process to separate the data based on the labels or outputs you've defined.
- When we look at the hyperplane in original input space it looks like a circle.





KERNEL FUNCTİONS

Let's look at an example:

$$\mathbf{x} = (x_1, x_2, x_3)^T$$
$$\mathbf{y} = (y_1, y_2, y_3)^T$$

Here x and y are two data points in 3 dimensions. Let's assume that we need to map x and y to 9-dimensional space. We need to do the following calculations to get the final result, which is just a scalar. The computational complexity, in this case, is $O(n^2)$.

$$\phi(\mathbf{x}) = (x_1^2, x_1x_2, x_1x_3, x_2x_1, x_2^2, x_2x_3, x_3x_1, x_3x_2, x_3^2)^T$$
$$\phi(\mathbf{y}) = (y_1^2, y_1y_2, y_1y_3, y_2y_1, y_2^2, y_2y_3, y_3y_1, y_3y_2, y_3^2)^T$$

$$\phi(\mathbf{x})^T\phi(\mathbf{y}) = \sum_{i,j=1}^{3} x_i x_j y_i y_j$$

KERNEL FUNCTİONS

However, if we use the kernel function, which is denoted as k(x, y), instead of doing the complicated computations in the 9-dimensional space, we reach the same result within the 3-dimensional space by calculating the dot product of x -transpose and y. The computational complexity, in this case, is O(n).

$$
\begin{aligned}
k(\mathbf{x}, \mathbf{y}) &= (\mathbf{x}^T \mathbf{y})^2 \\
&= (x_1 y_1 + x_2 y_2 + x_3 y_3)^2 \\
&= \sum_{i,j=1}^{3} x_i x_j y_i y_j
\end{aligned}
$$

In essence, what the kernel trick does for us is to offer a more efficient and less expensive way to transform data into higher dimensions. With that saying, the application of the kernel trick is not limited to the SVM algorithm. Any computations involving the dot products (x, y) can utilize the kernel trick.

EXAMPLES

For example let x and y be defined as $\mathbf{x = (x1, x2, x3)}$ and $\mathbf{y = (y1, y2, y3).}$

The mapping to 9 dimensions would be

$$\mathbf{f(x) = (x1x1, x1x2, x1x3, x2x1, x2x2, x2x3, x3x1, x3x2, x3x3)}$$

We can define a kernel which would be equivalent to the above equation

$$\mathbf{K(x, y) = dot(x, y) = x.y = (x^{T}y)^{2}}$$

Let,

$$\mathbf{x = (1, 2, 3)}$$
$$\mathbf{y = (4, 5, 6).}$$

Then:

$$\mathbf{f(x) = (1, 2, 3, 2, 4, 6, 3, 6, 9)}$$
$$\mathbf{f(y) = (16, 20, 24, 20, 25, 30, 24, 30, 36)}$$

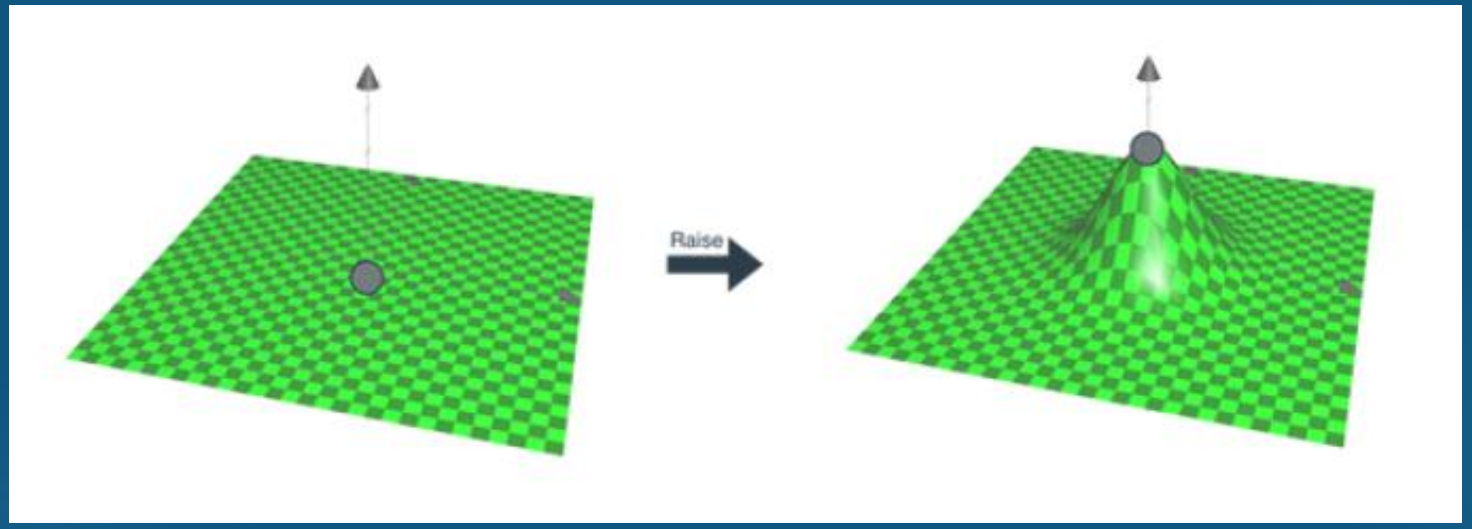
Calculating <f(x), f(y)> , gives us

$$\mathbf{16 + 40 + 72 + 40 + 100 + 180 + 72 + 180 + 324 = 1024}$$

Instead of doing so many calculations, if we apply the kernel instead:

$$\mathit{K(x, y) = (4 + 10 + 18)\ ^2 = 32^2 = 1024}$$

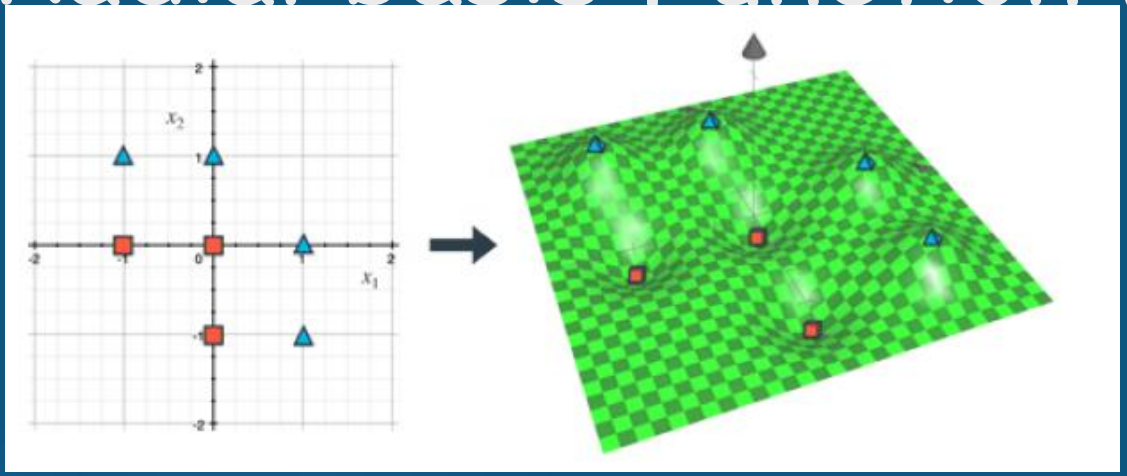
Radial basis Function (RBF) kernel

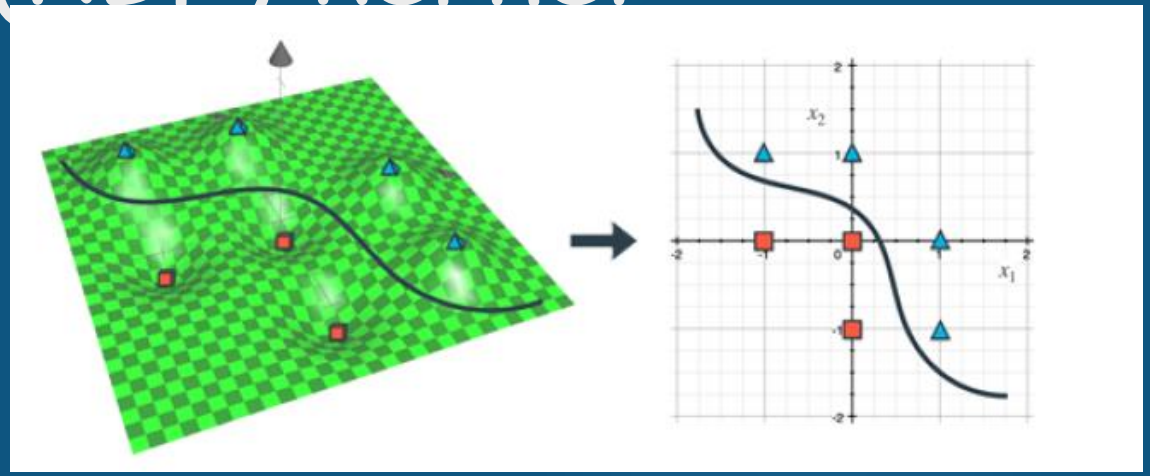


Imagine if you had a point in the plane, and the plane was like a blanket. Then you pinch the blanket at that point, and raise it. This is how a *radial basis function* looks like.

We can raise the blanket at any point we like, and that gives us one different radial basis function. The *radial basis function kernel* (also called rbf kernel) is precisely the set of all radial basis functions for every point in the plane.

Radial basis Function (RBF) kernel

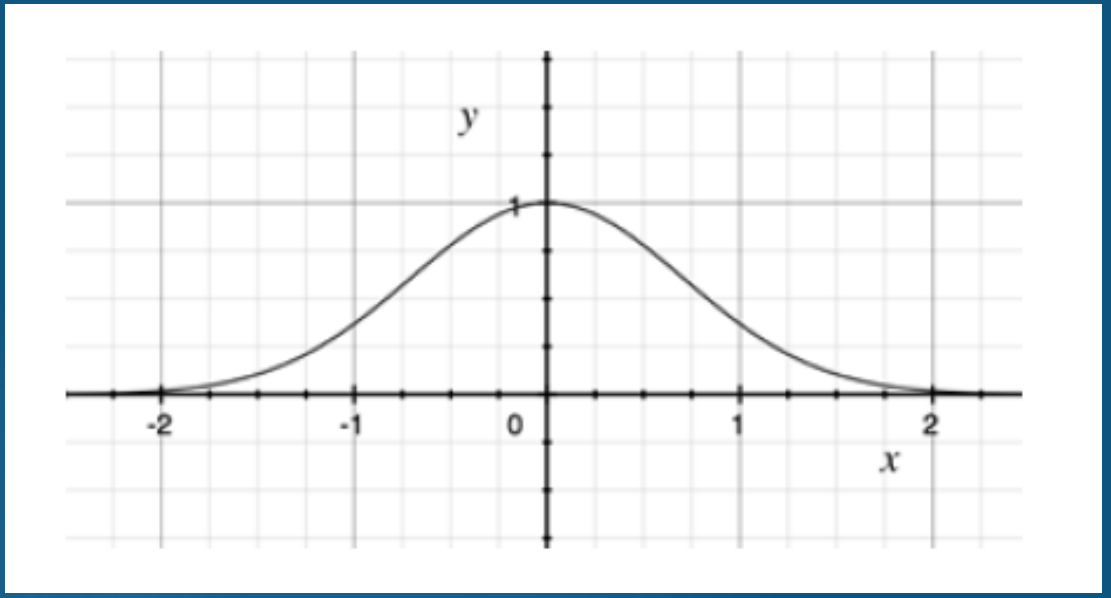




Lift the plane at every triangle, and push it down at every square. Then simply draw a plane at height 0, and intersect it with our surface. This is the same as looking at the curve formed by the points at height 0.

Imagine if there is a landscape with mountains and with the sea. The curve will correspond to the coastline, namely, where the water and the land meet. This gives us the curves (when we project everything back to the plane), and we obtain our desired classifier.

Radial Basis Function (RBF) kernel



1. One of the simplest radial basis function has the formula: $y = e^{-x^2}$

2. Notice that this bump happens at 0. If we wanted it at any different point, say p, we simply translate the formula to $y = e^{-(x-p)2}$

3. Thus, if we want to obtain the radial basis function centered at the point 5: $y = e^{-(x-5)2}$

Thank You